Graded groups. Some properties

1. Graded groups and their tensors

We need to set up some sign convention for this to work. For that we need the notion of a graded group.

A graded group G is just a group of the form

$$G = \bigoplus_{i=0}^{\infty} G_i.$$

Elements $g \in G_i$ are said to have *degree* i. We have the following notion of tensor of two graded groups A_* and B_*:

$$A_* \otimes B_* = \bigoplus_{k=0}^{\infty} \left(\bigoplus_{i=0}^{k} A_i \otimes B_{k-i} \right).$$

$(A_* \otimes B_*)_n = \oplus_{i=0} A_i \otimes B_{n-i}$ is *defined* to be the n-th graded component of $A_* \otimes B_*$.

We say that a map $f : C_* \to D_*$ between to graded groups is *homogeneous* of *degree* d if $f(C_i) \subset D_{i+d}$. We also introduce the following sign convention for tensor product of two graded maps $f_* : A_* \to C_*$ and $g_* : B_* \to D_*$ of degrees m and n:

$$(f \otimes g)(a \otimes b) = (-1)^{\deg(g)\deg(a)} f(a) \otimes g(b).$$

Easy exercise : Show that with the above sign convention, we have $(f \otimes g) \circ (f' \otimes g') = (-1)^{\deg(g)\deg(f')} (f \circ f') \otimes (g \circ g')$ for any four homogeneous maps f, f', g and g' such that $f \circ f'$ and $g \circ g'$ make sense.

2. $\times$ product and tensor product

We can think of $\Delta_*(X)$ as a graded group $\oplus_{i=0}^{\infty} \Delta_i(X)$. Thus we have a graded group $\Delta_*(X) \otimes \Delta_*(Y)$. We can make this into a chain complex by defining

$$\partial^{\Delta_*(X) \otimes \Delta_*(Y)} = \partial^{\Delta_*(X)} \otimes \mathrm{id}_{\Delta_*(Y)} + \mathrm{id}_{\Delta_*(X)} \otimes \partial^{\Delta_*(Y)},$$

which we write concisely as

$$\partial^{\otimes} = \partial^X \otimes \mathrm{id}_Y + \mathrm{id}_X \otimes \partial^Y,$$

where we think of the ∂ maps to have degree -1. Therefore, for $\sigma_p \in \Delta_p(X)$ and for $\tau_q \in \Delta_q(Y)$,

$$\begin{aligned}
\partial^{\otimes}(\sigma_p \otimes \tau_q) &= (\partial^X \otimes \mathrm{id}_Y + \mathrm{id}_X \otimes \partial^Y)(\sigma_p \otimes \tau_q) \\
&= (-1)^{\deg(\mathrm{id}_Y)\deg(\sigma_p)} \partial^X \sigma_p \otimes \tau_q + (-1)^{\deg(\partial)\deg(\sigma_p)} \sigma_p \otimes \partial^Y \tau_q \\
&= \partial^X \sigma_p \otimes \tau_q + (-1)^p \sigma_p \otimes \partial^Y \tau_q.
\end{aligned}$$

We shall use this formula quite often in the later lectures.

Now since $\times$ is bilinear, it induces a map $\times : \Delta_*(X) \otimes \Delta_*(Y) \to \Delta_*(X \times Y)$, which sends $a \otimes b$ to $a \times b$.

Proposition 1. *$\times : \Delta_*(X) \otimes \Delta_*(Y) \to \Delta_*(X \times Y)$ is a chain map.*

Proof.

$$\begin{aligned}\times(\partial^{\otimes}(a\otimes b)) &= \times(\partial a\otimes b+(-1)^{\deg a}a\otimes\partial b)\\ &= \partial a\times b+(-1)^{\deg a}a\times\partial b\\ &= \partial^{\times}(a\times b)\\ &= \partial^{\times}(\times(a\otimes b))\end{aligned}$$

for all $a\in\Delta_*(X)$ and $b\in\Delta_*(Y)$. Therefore, $\times$ is a chain map. □

3. Toward's Künneth

Our final aim is to understand $\Delta_*(X\times Y)$. To do that we shall prove that as a chain complex it is homotopy equivalent to $\Delta_*(X)\otimes\Delta_*(Y)$. We have already constructed a map

$$\times:\Delta_*(X)\otimes\Delta_*(Y)\to\Delta_*(X\times Y).$$

All we need is a map, Θ, in the opposite direction which will give the "homotopy inverse", that is $\Theta\circ\times$ and $\times\circ\Theta$ are homotopic to identities on the respective spaces.

The construction will again use the methods of acyclic models. To help the induction process we need the following proposition.

Definition 2. For a space T with basepoint t_0, the *augmentation* map ϵ_*^T : $\Delta_*(T)\to\Delta_*(T)$ is given by $\epsilon_p^T=0$ for $p\neq 0$ and $\epsilon_0^T(\sum n_t t)=(\sum n_t)t_0$.

Proposition 3. *Suppose X and Y are contractible spaces, and let $x_0\in X$ and $y_0\in Y$ be basepoints. Then $\Delta_*(X)\otimes\Delta_*(Y)$ is chain contractible. That is the tensor product of the augmentations maps $\epsilon^X\otimes\epsilon^Y$ is homotopic to the identity map* $\mathrm{id}_{\Delta_*(X)}\times\mathrm{id}_{\Delta_*(Y)}$. *Consequently,*

$$H^n(\Delta_*(X)\otimes\Delta_*(Y))=\begin{cases}0 & \text{if } n>0\\ \mathbb{Z}(x_0\otimes y_0) & \text{for } n=0.\end{cases}$$

Proof. Since X is contractible, one can construct a homotopy $D:\Delta_*(X)\to\Delta_{*+1}(X)$ between ϵ_*^X and $\mathrm{id}_{\Delta_*(X)}$. Define $E=D\times\mathrm{id}_{\Delta_*(Y)}+\epsilon\times D$.

Reading exercise: Check that this is the homotopy we need. (Bredon, Page 316, Lemma 1.1). □