Cross product on simplices

Today we shall define a bilinear map

$$\times : \Delta_p(X) \times \Delta_q(Y) \longrightarrow \Delta_{p+q}(X).$$

This will also be the first demonstration of the method of acyclic models which we shall see a lot in the next few lectures. It is an inductive procedure, used along with naturality conditions in the given situation. Let us see the first example.

1. Existence of cross product

Notation 1. Let τ_x be the singular 0 simplex defined by the point x: $\tau_x : e_0 \to X$ with $\tau(e_0) = x$. For $\sigma : \Delta_q \to Y$, we let $\tau_x \times \sigma : \Delta_q \to X \times Y$ to be the map $w \mapsto (x, \sigma(w)$. Similarly, in a symmetric fashion, $\gamma \times \tau_y : w \mapsto (\gamma(w), y)$ for $\gamma \in \Delta_p(X)$. These give us maps $\Delta_0(X) \otimes \Delta_q(Y) \to \Delta_q(X \times Y)$ and $\Delta_p(X) \times \Delta_0(Y) \to \Delta_p(X \times Y)$ respectively. We take these as definitions of cross product, in the cases when at least one of the simplices involved is zero dimensional.

Theorem 2. *There exist bilinear maps $\times : \Delta_p(X) \times \Delta_q(Y) \to \Delta_{p+q}(X \times Y)$ such that*

(1) *For $x \in X$, $y \in Y$ and $\sigma \in \Delta_p(X)$, $\gamma \in \Delta_q(Y)$, $\tau_x \times \gamma$ and $\sigma \times \tau_y$ are defined as above.*
(2) *If $f : X \to X'$ and $g : Y \to Y'$ are maps; and suppose the product map is denoted by $\langle f, g \rangle : X \times Y \to X' \times Y'$; then*
$$\langle f, g \rangle_\Delta (a \times b) = f_\Delta(a) \times g_\Delta(b);$$
(3) $\partial(a \times b) = \partial a \times b + (-1)^{\deg a} a \times \partial b.$

Proof. By the given conditions, we already know what the $\times$ product is, when either $p = 0$ or $q = 0$. Assume that we know the $\times$ product for $p + q < k$. Note that this means that $k \geq 2$.

Suppose $p + q = k$. We want to define $\iota_p \times \iota_q$ where $\iota_r : \Delta_r \to \Delta_r$ is the identity map. (These are our acyclic models. The nomenclature will be clear in a few paragraphs.)

Let $B = \partial \iota_p \times \iota_q + (-1)^p \partial \iota_p \times \iota_q$. Then, using the boundary formula for the $\times$ product for $p+q < k$, we get $\partial B = 0$ (Check.) Therefore, B is a cycle on $\Delta_p \times \Delta_q$, which is contractible, and hence have all homologies zeroes, except maybe for H_0. Since $k \geq 2$, we have that B is a boundary. Choose $\iota_p \times \iota_q \in \Delta_{p+q}(\Delta_p \times \Delta_q)$ such that $\partial(\iota_p \times \iota_q) = B = \partial \iota_p \times \iota_q + (-1)^p \partial \iota_p \times \iota_q$.

Fix p and q as above satisfying $p + q = k$. Now, let $\sigma \in \Delta_p(X)$ and $\theta \in \Delta_q(Y)$. We want to define $\sigma \times \theta$. Consider the product map $\langle \sigma, \theta \rangle : \Delta_p \times \Delta_q \to X \times Y$. Define

$$\sigma \times \theta = \langle \sigma, \theta \rangle_\Delta (\iota_p \times \iota_q).$$

We only have to check that this definition satisfies naturality and the boundary condition. We do these in order.

Proof of naturality. Let $f : X \to X'$ and $g : Y \to Y'$ be two maps. Let $\sigma \in \Delta_p(X)$ and $\theta \in \Delta_q(Y)$. We want to compute $\langle f, g \rangle_\Delta (\sigma \times \theta)$.

$$\begin{aligned} \langle f, g \rangle_\Delta (\sigma \times \theta) &= \langle f, g \rangle_\Delta \langle \sigma, \theta \rangle_\Delta (\iota_p \times \iota_q) = (\langle f, g \rangle \circ \langle \sigma, \theta \rangle)_\Delta (\iota_p \times \iota_q) \\ &= \langle f \circ \sigma, g \circ \theta \rangle_\Delta (\iota_p \times \iota_q) = (f \circ \sigma) \times (g \circ \theta) \\ &= f_\Delta(\sigma) \times g_\Delta(\theta). \end{aligned}$$

proving naturality.

Now we shall use naturality to prove the boundary formula.

Proof of boundary formula. Let $\sigma \in \Delta_p(X)$ and $\theta \in \Delta_q(Y)$ as above. Then

$$\begin{aligned}
\partial(\sigma \times \theta) &= \partial\langle \sigma, \theta\rangle_\Delta(\iota_p \times \iota_q) = \langle \sigma, \theta\rangle_\Delta \partial(\iota_p \times \iota_q) \\
&= \langle \sigma, \theta\rangle_\Delta(\partial\iota_p \times \iota_q + (-1)^p \iota_p \times \partial\iota_q) \\
&= \langle \sigma, \theta\rangle_\Delta(\partial\iota_p \times \iota_q) + \langle \sigma, \theta\rangle_\Delta((-1)^p \iota_p \times \partial\iota_q) \\
&= (\sigma_\Delta(\partial\iota_p) \times \theta_\Delta \iota_q) + (-1)^p(\sigma_\Delta \iota_p \times \theta_\Delta(\partial\iota_q)) \\
&= (\partial(\sigma_\Delta \iota_p) \times \theta_\Delta \iota_q) + (-1)^p(\sigma_\Delta \iota_p \times \partial(\theta_\Delta \iota_q)) \\
&= \partial\sigma \times \theta + (-1)^p \sigma \times \partial\theta
\end{aligned}$$

as was to be proved. □

2. Cross product induces maps between homologies

Definition 3. If (X, A) and (Y, B) are pairs of spaces then $(X, A) \times (Y, B)$ denotes the pair $(X \times Y, (X \times B) \cup (A \times Y))$.

Proposition 4. *The cross product induces a map*

$$\times : H_p(X, A) \times H_q(Y, B) \to H_{p+q}((X, A) \times (Y, B))$$

defined by $[\![a]\!] \times [\![b]\!] = [\![a \times b]\!]$, *where* $[\![\sigma]\!]$ *is the class of the simplex* σ *in the corresponding homology group.*

Proof. Note that $a \in Z_p(X, A)$ if (by definition) $\partial a \in \Delta_{p-1}(A)$. Suppose $a \in Z_p(X, A)$ and $b \in Z_q(Y, B)$. Then

$$\partial(a \times b) = \partial a \times b + (-1)^p a \times \partial b$$

which is in $\Delta_{p+q-1}(X \times B \cup A \times Y)$.

Now suppose a and B are as above. Also let $c \in \Delta_{p+1}(X)$ be such that $\partial c \in \Delta_p(A)$ and $d \in \Delta_{q+1}(Y)$ be such that $\partial d \in \Delta_q(B)$. Then

$$\begin{aligned}
(a + \partial c) \times (b + \partial d) &= a \times b + a \times \partial d + \partial c \times b + \partial c \times \partial d \\
&= a \times b + (-1)^p \partial(a \times d) + (-1)^{p+1} \partial a \times d + \partial(c \times b) + \\
&\qquad (-1)^p c \times \partial b + \partial(c \times \partial d) \\
&= a \times b + \partial((-1)^p a \times d + c \times d + c \times \partial d) + \\
&\qquad ((-1)^{p+1} \partial a \times d + (-1)^p c \times \partial b) \\
&= a \times b + \text{ a boundary } + \text{ chain in } A \times Y \cup X \times B
\end{aligned}$$

and hence $[\![a \times b]\!] = [\![(a + \partial c) \times (b + \partial d)]\!]$.

Thus $\times$ induces a map between homologies. □