Some corollaries to the UCT

Corollary 1. *If $H_{n-1}(X, A)$ and $H_n(X, A)$ are finitely generated then so is $H^n(X, A; Z)$. Moreover,*

$$H^n(X, A; \mathbb{Z}) \cong F_n \oplus T_{n-1}$$

where F_n is the free part of $H_n(X, A)$ and T_n is the torsion part.

Proof. Suppose T is a torsion abelian group. Consider the injective resolution of $\mathbb{Z}$ given by $0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Then Ext is computed as the cohomology of the sequence $0 \to \hom(A, \mathbb{Q}) \to \hom(A, \mathbb{Q}/\mathbb{Z}) \to 0$. Since $\hom(A, \mathbb{Q}) = 0$, $\operatorname{Ext}^1(A, \mathbb{Z}) = \hom(A, \mathbb{Q}/\mathbb{Z}) \cong A$ (one can see this using classification of abelian groups, for example). Also since a free abelian group F is projective, $\operatorname{Ext}^1(F, \mathbb{Z}) = 0$.

On the other hand, $\hom(F, \mathbb{Z}) \cong F$ and $\hom(T, \mathbb{Z}) = 0$. This gives us from UCT-I,

$$H^n(X, A; \mathbb{Z}) = \operatorname{Ext}^1(H_{n-1}(X, A), \mathbb{Z}) \oplus \hom(H_n(X), \mathbb{Z}) = T_{n-1} \oplus F_n,$$

as was to be proved. □

Corollary 2. $H_n(X, A; \mathbb{Q}/\mathbb{Z}) = H_n(X, A) \otimes \mathbb{Q}/\mathbb{Z} \oplus T_{n-1}$.

Proof. The proof follows from UCT-II, once we prove that $\operatorname{Tor}_1(A, \mathbb{Q}/\mathbb{Z}) \cong TA$ where TA is the torsion part of A. Note that the short exact sequence $0 \to TA \to A \to A/TA \to 0$ yields $0 \to \operatorname{Tor}_1(TA, B) \to \operatorname{Tor}_1(A, B) \to \operatorname{Tor}_1(A/TA, B) = 0$ for all B, since A/TA is torsion free. Therefore $\operatorname{Tor}_1(TA, B) \cong \operatorname{Tor}_1(A, B)$ for all abelian groups A and B.

Also from the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, we get $0 = \operatorname{Tor}_1(TA, Q) \to \operatorname{Tor}_1(TA, \mathbb{Q}/\mathbb{Z}) \to TA \otimes \mathbb{Z} \to TA \otimes Q = 0$. Therefore, $TA \cong TA \otimes \mathbb{Z} \cong \operatorname{Tor}_1(TA, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Tor}_1(A, \mathbb{Q}/\mathbb{Z})$ as was to be proved. □

Corollary 3. *Suppose $\varphi : A_* \to B_*$ be a chain map of free chain complexes inducing isomorphisms $\varphi_* : H_n(A_*) \to H_n(B_*)$ for all n. Then the induced maps $(\varphi \otimes 1)_* : H_n(A_* \otimes G) \to H_n(B_* \otimes G)$ and $\varphi^* : H^n(\hom(B_*, G)) \to H^n(\hom(A_*, G))$ are isomorphisms for all n.*

Proof. The proof follows from the functoriality in the two UCTs and five lemma. □

This immediately yields

Corollary 4. *If $\varphi : (X, A) \to (Y, B)$ be such that $\varphi_* : H_n(X, A) \to H_n(Y, B)$ are isomorphisms for all n, then*

$$\varphi_* : H_n(X, A; G) \to H_n(Y, B; G), \text{ and}$$
$$\varphi^* : H^n(Y, B; G) \to H^n(X, A; G)$$

are isomorphisms for all n and all G.