Example showing that the splitting is not functorial

Suppose S^2 is the two sphere which is obtained by identifying the boundary of a closed disc. P^2 be the projective plane obtained by identifying the antipodal points of the boundary of a closed disc.

We have a quotient map $\varphi : P^2 \to S^2$ obtained by identifying the boundary (of the image of the open disc) in P^2.

Recall that

$$H_1(P^2) = \frac{\mathbb{Z}}{2\mathbb{Z}}, \qquad H_2(P^2) = 0,$$
$$H_1(S^2) = 0, \text{ and} \qquad H_2(S^2) = \mathbb{Z}.$$

This implies that $\varphi_* : H_i(P^2) \to H_i(S^2)$ is the zero map for $i = 1, 2$. Thus if the splitting in the UCT-II,

$$H_2\left(X; \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \cong \left(H_2(X) \otimes \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \oplus \mathrm{Tor}_1\left(H_1(X), \frac{\mathbb{Z}}{2\mathbb{Z}}\right),$$

was natural, the induced map $\varphi_* : H_2(P^2; \mathbb{Z}/2\mathbb{Z}) \to H_2(S^2, \mathbb{Z}/2\mathbb{Z})$ would be zero. However, we claim this is not true.

Let σ and τ be the unique two cells of P^2 and S^2 respectively. Since singular homology is computed with CW homology (Cellular approximation theorem), φ_* can be computed using $\varphi_\Delta : C_2(P^2, \mathbb{Z}/2\mathbb{Z}) \to C_2(S^2, \mathbb{Z}/2\mathbb{Z})$. Furthermore, we know that $\varphi_\Delta(\sigma) = \mathrm{sgn}(\varphi_{\tau,\sigma})\tau = \pm\tau$. Since σ and τ are the only 2 cells in P^2 and S^2, H_2 is computed by $\varphi_{\tau,\sigma}$. Therefore, φ_Δ is an isomorphism and hence so is φ_*. This is a contradiction.