Universal coefficient theorems

Now we have all the prerequisites to state and prove the universal coefficient theorem (UCT). The main content of the theorem is that usual homology is the finest invariant. There are two versions of UCT. The first one will give a relationship between cohomology with arbitrary coefficients to singular homology (with $\mathbb{Z}$ coefficients). The second version of UCT relates homology with coefficients to homology without coefficients (i.e. with $\mathbb{Z}$ coefficient).

1. UCT-I

First we give a homological algebra version of the theorem from which the main conclusion will follow as an easy corollary.

Theorem 1. *Let C_* be a chain complex of free abelian groups. Then for any abelian group G, we have a short exact sequence*

$$0 \to \operatorname{Ext}^1(H_{n-1}(C_*), G) \xrightarrow{\alpha} H^n(\hom(C_*, G)) \xrightarrow{\beta} \hom(H_n(C_*), G) \to 0.$$

which is functorial in both C_* and G. Furthermore, the sequence splits and the splitting is functiorial in G but not in C_*.

Proof. Given a complex $C_* := \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$, one defines

$$Z_n = \ker \partial_n \subset C_n,$$
$$B_n = \operatorname{im} \partial_{n+1} \subset C_n.$$

Then we have the following short exact sequences

$$0 \to Z_n \to C_n \to B_{n-1} \to 0,$$
$$0 \to B_n \to Z_n \to H_n \to 0.$$

Now using the long exact sequence of Ext on these two short exact sequences we get a diagram

$$\begin{array}{ccccc}
\hom(Z_{n-1}, G) & \overset{h}{\twoheadleftarrow} & \hom(C_{n-1}, G) & & \hom(H_n, G) \\
\downarrow f & & \downarrow \delta^{n-1} & & \downarrow e \\
\hom(B_{n-1}, G) & \overset{a}{\rightarrowtail} & \hom(C_n, G) & \overset{b}{\twoheadrightarrow} & \hom(Z_n, G) \\
\downarrow g & & \downarrow \delta^n & & \downarrow c \\
\operatorname{Ext}^1(H_{n-1}, G) & & \hom(C_{n+1}, G) & \overset{d}{\leftarrowtail} & \hom(B_n, G)
\end{array}$$

Note that Z_i and B_i are free and thus have $\operatorname{Ext}^i(\ ,\ ) = 0$ with any abelian group. Using this fact and using the long exact sequence of Ext, we get the injectivity of a, e and d and surjectivity of b, h and g. Also the two square are commutative. (Check!)

Now for $x \in \hom(B_{n-1}, G)$, look at $a(x)$. $\delta^n(a(x)) = d \circ c \circ b \circ a(x) = 0$. Thus a induces a map $\hom(B_{n-1}, G) \xrightarrow{\tilde{a}} \ker \delta^n / \operatorname{im} \delta^{n-1} = H^n(\hom(C_*, G))$. Also for an element $f(y) \in \hom(B_{n-1}, G)$, $a \circ f(y) = a \circ f \circ h(z)$, for some z, which exists since h is surjective. Therefore $a \circ f(y) = \delta^{n-1}(z) \in \operatorname{im} \delta^{n-1}$. Therefore, $a \circ f(y) \mapsto 0 \in H^n(\hom(C_*, G))$. In other words, we have a map

$$\bar{a} : \frac{\hom(B_{n-1}, G)}{\operatorname{im} f} \cong \operatorname{Ext}^1(H_{n-1}, G) \to H^n(\hom(C_*, G)).$$

Now we define a map $H^n(\hom(C_*, G)) \to \hom(H^n(C_*), G)$. For that note that if $u \in \ker \delta^n$, then $d \circ c \circ b(u) = \delta^n(u) = 0$. Since d is injective, we get $c \circ b(u) = 0$. Therefore, $b(u) \in \ker c = \operatorname{im} e$. Since e is injective there exists a unique $v \in \hom(H_n, G)$, such that $b(u) = e(v)$. Define a map $\tilde{b} : \ker \delta^n \to \hom(H_n, G)$ by $\tilde{b}(u) = v$. Now $b(\delta^{n-1}(w)) = b \circ a \circ f \circ h(w) = 0$. Therefore, by the definition of $\tilde{b}$, $\tilde{b}(\delta^{n-1}(w)) = 0$. Thus, we get a map

$$\bar{b} : \frac{\ker \delta^n}{\operatorname{im} \delta^{n-1}} \cong H^n(\hom(C_*, G)) \to \hom(H_n, G).$$

Now that we have the maps in place, we do the usual checks to prove that the given sequence is a short exact sequence :

$\bar{a}$ **is injective:** Suppose $\bar{a}(x) = 0$. Then $\tilde{a}(x') = 0$ where $x' \in \hom(B_{n-1}, G)$ such that $g(x') = x$. Now $\tilde{a}(x') = a(x') \mod \operatorname{im} \delta^{n-1}$. Thus $\bar{a}(x') = 0$ would translate to $a(x') \in \operatorname{im} \delta^{n-1}$. Thus $a(x') = \delta^{n-1}(t)$ for some t. Thus $a(x') = a \circ f \circ h(t)$. Since a is injective, $x' = f \circ h(t)$. Therefore, $x = g(x') = g \circ f \circ h(t) = 0$, as was to be proved.

$\bar{b}$ **is surjective:** Let φ be an element of $\hom(H_n, G)$. Consider $e(\varphi)$. Since b is surjective, there exists $\psi \in \hom(C_n, G)$ such that $b(\psi) = e(\varphi)$. Note that $(d \circ c \circ b)(\psi) = d \circ c \circ e(\varphi) = 0$. Therefore, $\delta^n(\psi) = 0$ and hence $\tilde{b}(\psi) = \varphi$. From this it follows that $\bar{b}(\psi) = \varphi$, proving the fact that $\bar{b}$ is surjective.

$\bar{b} \circ \bar{a} = 0$**:** For $\gamma \in \operatorname{Ext}^1(H_{n-1}, G)$, Let γ' be a lift of γ in $\hom(B_{n-1}, G)$, i.e., $g(\gamma') = \gamma$. $\bar{b} \circ \bar{a}(\gamma) = \bar{b}(a(\gamma) \mod \operatorname{im} \delta^{n-1}) = b(a(\gamma)) = 0$.

$\ker \bar{b} \subset \operatorname{im} \bar{a}$**:** Check that $\operatorname{im} \bar{a} = \operatorname{im} a \mod \operatorname{im} \delta^{n-1}$ and similarly $\ker \bar{b} = \ker b \mod \operatorname{im} \delta^{n-1}$. Now the result follows from the fact that $\ker b = \operatorname{im} a$. Actually, this would have also proved that $\bar{b} \circ \bar{a} = 0$.

Now to check functoriality, just notice that all the constructions above are functorial. Check this!

The splitting is given by the fact that the sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$ being a short exact sequence of free abelian groups, splits. Use that splitting to construct a splitting $\hom(C_n, G) \to \hom(B_{n-1}, G)$. Check that this map induces a map $H^n(\hom(C_*, G)) \to \operatorname{Ext}^1(H_{n-1}, G)$ which provides the required splitting. Again note that this contruction is functiorial in G, but may not be in C_*. □

Recall that

Definition 2. *Singular cohomology* of (X, A) with coefficients in G is defined as $H^n(X, A; G) = H^n(\hom(\Delta_*(X, A), G))$.

Corollary 3. *Now take C_* to be the singular complex $\Delta_*(X, A)$. Then we have the short exact sequence*

$$0 \to \operatorname{Ext}^1(H_{n-1}(X, A), G) \to H^n(X, A; G) \to \hom(H_n(X, A), G) \to 0$$

which is functorial in both (X, A) and G. Moreover the short exact sequence splits, where the splitting is functorial in G but not in (X, A).