Ext and Tor of abelian groups

1. For abelian groups higher Ext and Tor are zero.

Recall that an abelian group is injective if and only if it is divisible. Divisible groups have the property that quotients of divisible groups are also divisible. Let A be any abelian group. We know that there exists an injective map from A to some divisible group $D(A)$,

$$j : A \hookrightarrow D(A).$$

Consider the exact sequence

$$0 \to A \to D(A) \to \frac{D(A)}{A} \to 0.$$

Here $D(A)/A$ being a quotient of a divisible group is also divisible, and hence injective. Therefore $0 \to D(A) \to D(A)/A \to 0 \to \cdots$ is an injective resolution of A. Therefore, $\mathrm{Ext}^i(B, A) = 0$ for all abelian groups B and for all $i \geq 2$. Since A was an abelian group, we have

Proposition 1. $\mathrm{Ext}^i(A, B) = 0$ *for all* $i \geq 2$ *and all abelian groups* A *and* B*.*

A similar phenomenon happens for projective (even free) resolutions of abelian groups. Consider an abelian group A. Consider any set of generators of A and consider the free abelian group $F(A)$ generated by those generators. Then the canonical map $F(A) \to A$ is surjective. Let K be the kernel. K, a subgroup of a free group, is free. K is abelian being a subgroup of an abelian group $F(A)$. Note K and $F(A)$, being free abelian, are also projective abelian groups. Thus the projective resolution of A is given by

$$\cdots \to 0 \to K \to F(A) \to 0.$$

From this we conclude the following proposition.

Proposition 2.

$$\mathrm{Tor}_i(A, B) = 0 \text{ for all abelian groups } A \text{ and } B, \text{ and } i \geq 2.$$

Note that this means that for abelian groups, the long exact sequences mentioned in the previous lecture will only have six terms:

$$0 \to \hom(M, A) \to \hom(M, B) \to \hom(M, C) \to \\ \mathrm{Ext}^1(M, A) \to \mathrm{Ext}^1(M, B) \to \mathrm{Ext}^1(M, C) \to 0$$

$$0 \to \hom(M'', N) \to \hom(M, N) \to \hom(M', N) \to \\ \mathrm{Ext}^1(M'', N) \to \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(M', N) \to 0$$

and similar sequences for Tor.

2. Understanding torsion

Theorem 3. *An abelian group* A *is torsion free if and only if for any abelian group* B*,* $\mathrm{Tor}_1(A, B) = 0$*.*

Proof. Suppose A has an element x such that $nx = x + \cdots + x = 0$. Now consider the short exact sequence $0 \to n\mathbb{Z} \hookrightarrow \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$. Since both $\mathbb{Z}$ and $n\mathbb{Z}$ being finitely generated and free, they are projective. Hence $\mathrm{Tor}_i(\mathbb{Z}, A) = 0$ and $\mathrm{Tor}_i(n\mathbb{Z}, A) = 0$ for all $i \geq 1$. Hence we have an exact sequence of Tor's,

$$0 \to \mathrm{Tor}_1(\mathbb{Z}/n\mathbb{Z}, A) \to (n\mathbb{Z}) \otimes A \to \mathbb{Z} \otimes A \to \mathbb{Z}/n\mathbb{Z} \otimes A \to 0.$$

Note that $n \otimes x \neq 0 \in (n\mathbb{Z}) \otimes A$ maps to $0 \in \mathbb{Z} \otimes A$ and hence lies in the image of $\mathrm{Tor}_1(\mathbb{Z}/n\mathbb{Z}, A)$, which implies that $\mathrm{Tor}_1(\mathbb{Z}/n\mathbb{Z}, A) \neq 0$. This proves that if $\mathrm{Tor}_1(B, A) = 0$ for all B, then A is torsion free.

Now suppose A is torsion free, but there exists a B such that $\mathrm{Tor}_1(A, B) \neq 0$. We shall arrive at a contradiction from here. Let $0 \to K \to F \to B \to 0$ be a short exact sequence with F and K free. Then consider the following diagram.

$$0 \to \mathrm{Tor}_1(B, A) \xrightarrow{\delta} K \otimes A \xrightarrow{\iota} F \otimes A \xrightarrow{\pi} B \otimes A \to 0.$$

Let $y \neq 0 \in \mathrm{Tor}_1(B, A)$. Let $\delta(y) = \sum_{i=1}^{n} k_i \otimes a_i$. Let G be the finitely generated subgroup of A generated by $a_1, \ldots, a_n$. Note that $\delta(y) \in G \otimes K$ by definition of G. We have a short exact sequence $0 \to G \to A \to A/G \to 0$. This allows us to draw the following diagram

$$\begin{array}{ccccccc}
 & & \mathrm{Tor}_1(A/G, K) & & \mathrm{Tor}_1(A/G, F) & & \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\
0 \to \mathrm{Tor}_1(G, B) & \xrightarrow{\delta'} & G \otimes K & \xrightarrow{\iota'} & G \otimes F & \xrightarrow{\pi'} & G \otimes B \to 0 \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i \otimes K} & & \downarrow{\scriptstyle i \otimes F} & & \downarrow{\scriptstyle i \otimes B} \\
0 \to \mathrm{Tor}_1(A, B) & \xrightarrow{\delta} & A \otimes K & \xrightarrow{\iota} & A \otimes F & \xrightarrow{\pi} & A \otimes B \to 0
\end{array}$$

For the sake of clarity, $\sum k_i \otimes a_i \in G \otimes K$ be denoted by a. Then $(i \otimes K)(a) = \delta(y)$. Consider $\iota'(a)$.

$$\begin{aligned}(i \otimes F)(\iota'(a)) &= \iota(i \otimes K(a)) = \iota(\delta(y)) \\ &= 0.\end{aligned}$$

This means that $\iota'(a) \in \ker(i \otimes F) = \operatorname{im}\beta$. However, since F is free and hence projective, $\mathrm{Tor}_1(A/G, F) = 0$ and hence $\iota'(a) = 0$. This means that $a \in \ker \iota' = \operatorname{im}\delta'$. Since $a \neq 0$, there exists $b \neq 0 \in \mathrm{Tor}_1(G, B)$ such that $a = \delta'(b)$.

However, G is a subgroup of A and hence is torsion-free abelian. By construction G is finitely generated. Now any finitely generated, torsion free, abelian group is a free abelian group and hence is projective. Therefore, $\mathrm{Tor}_1(G, B) = 0$ contradicting the fact that $b \in \mathrm{Tor}_1(G, B)$ is nonzero. □

Exercise 4.
(1) Prove that P is projective iff for every short exact sequence $0 \to A \to B \to C \to 0$, $0 \to \hom(P, A) \to \hom(P, B) \to \hom(P, C) \to 0$ is exact.
(2) Prove that P is projective iff $\mathrm{Ext}^1(P, G) = 0$ for all G.
(3) Prove that I is injective iff for every short exact sequence $0 \to A \to B \to C \to 0$, $0 \to \hom(C, I) \to \hom(B, I) \to \hom(A, I) \to 0$ is exact.
(4) Prove that I is injective iff $\mathrm{Ext}^1(G, I) = 0$ for all G.