Injective resolutions, Chain homotopy, Ext

Let R be a commutative ring with 1 as usual.

In the last lecture we saw that given any R module M, we can put it inside an injective module $I(M)$. In this lecture we shall construct an *injective resolution* using this fact. Then we prove that any two injective resolutions are chain homotopic. Using this fact, we define Ext groups.

1. Injective resolutions

Start with any R module M. Then we know that M admits an injective map to an injective module I^0.

$$0 \to M \xrightarrow{i_M} I^0.$$

Consider the quotient map $p^0 : I^0 \to \operatorname{coker} i_M$, and let I^1 be an injective module such that we have an injective morphism $i^0 : \operatorname{coker} i_M \to I^1$. Let $d^0 = i^0 \circ p^0$. Then we get an exact sequence

$$0 \to M \xrightarrow{i_M} I^0 \xrightarrow{d^0} I^1.$$

Now suppose we have chosen $I^0, \ldots, I^m$ along with $d^0, \ldots, d^{m-1}$. As before consider the quotient map $p^m : I^m \to \operatorname{coker} d^{m-1}$. Let $i^m : \operatorname{coker} d^{m-1} \to I^{m+1}$ be an injective map into an injective module I^{m+1}. As before define $d^m = i^m \circ p^m : I^m \to I^{m+1}$. Clearly, since i^m is injective, $\ker d^m = \ker p^m = \operatorname{im} d^{m-1}$. This gives us the following exact sequence (with infinitely many terms)

$$0 \to M \xrightarrow{i_M} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{m-1}} I^m \xrightarrow{d^m} I^{m+1} \xrightarrow{d^{m+1}} \cdots .$$

We shall call $I^* := 0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{m-1}} I^m \xrightarrow{d^m} \cdots$ to be <u>an</u> *injective resolution* of M. Note that all the terms in this complex are injective, and moreover $H^0(I^*) = M$ and $H^i(I^*) = 0$ for all $i \geq 1$.

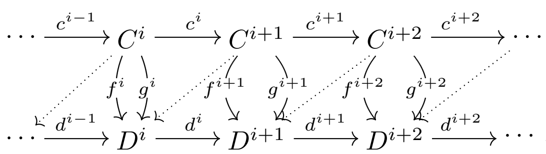
Observe that an injective resolution is not at all unique. At each step of the construction, we have a choice of an injective module containing the cokernel of the previous map. However, things are not too bad as explained in the next section.

2. Chain homotopy

Let $C^* := \cdots \xrightarrow{c^{i-1}} C^i \xrightarrow{c^i} C^{i+1} \xrightarrow{c^{i+1}} \cdots$ and $D^* := \cdots \xrightarrow{d^{i-1}} D^i \xrightarrow{d^i} D^{i+1} \xrightarrow{d^{i+1}} \cdots$ be two cochain complexes. Let $f^* : C^* \to D^*$ be a cochain map. Recall that this means that all the squares in the following diagram commute

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{c^{i-1}} & C^i & \xrightarrow{c^i} & C^{i+1} & \xrightarrow{c^{i+1}} & C^{i+2} & \xrightarrow{c^{i+2}} & \cdots \\
 & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \downarrow{\scriptstyle f^{i+2}} & & \\
\cdots & \xrightarrow{d^{i-1}} & D^i & \xrightarrow{d^i} & D^{i+1} & \xrightarrow{d^{i+1}} & D^{i+2} & \xrightarrow{d^{i+2}} & \cdots .
\end{array}$$

Two such maps $f^*, g^* : C^* \to D^*$ are said to be *chain homotopic* if there exists maps $h^i : C^i \to D^{i-1}$, $i \in \mathbb{Z}$, as in the following diagram



where the dotted arrows are h^i, h^{i+1}, h^{i+2} and h^{i+3} respectively from left to right, such that

$$f^i - g^i = d^{i-1} \circ h^i + h^{i+1} \circ c^i \text{ for all } i \in \mathbb{Z}.$$

Proposition 1. *Let M and N be two R modules. Suppose $f : M \to N$ is an R-module homomorphism. Fix injective resolutions I^* and J^* of M and N respectively. Then f can be "extended" to a cochain map $f^* : I^* \to J^*$. Moreover, any two such extensions are chain homotopic.*

Proof. Consider the maps $i_M : M \hookrightarrow I^0$, and $i_N \circ f : M \to N \to J^0$. The fit into a diagram with exact top row:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{i_M} & I^0 \\ & & {\scriptstyle i_N \circ f}\downarrow & \swarrow {\scriptstyle f^0} & \\ & & J^0 & & \end{array},$$

where the map indicated by the dotted arrow, f^0 exists by the universal property of the injective module J^0. Now consider the maps $d_J^0 : J^0 \to J^1$ and $d_I^0 : I^0 \to I^1$. Note that $d_J^0 \circ f^0 \circ i_M = d_J^0 \circ i_N \circ f = 0$. Therefore, we get a map $\widehat{d_J^0 \circ f^0}$: $\operatorname{coker} i_M \to J^1$, such that $\widehat{d_J^0 \circ f^0} \circ p^0 = d_J^0 \circ f^0 : I^0 \to J^1$, where $p^0 : I^0 \to \operatorname{coker} i_M$ is the quotient map.

Now note that $\operatorname{coker} i_M \cong \operatorname{im} d_I^1$. To see this, observe that $\operatorname{im} i_M = \ker d_I^1$. Therefore, by third isomorphism theorem, $\operatorname{im} d_I^1 \cong I_0 / \ker d_I^1 = I^0 / \operatorname{im} i_M$ which is, by definition, $\operatorname{coker} i_M$. Therefore the composite map $\alpha^0 : \operatorname{coker} i_M \xrightarrow{\cong} \operatorname{im} d_I^1 \hookrightarrow I_1$ is injective. Hence we can make a diagram with exact top row as before:

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{coker} i_M & \xrightarrow{\alpha^0} & I_1 \\ & & {\scriptstyle \widehat{d_J^0 \circ f^0}}\downarrow & \swarrow {\scriptstyle f^1} & \\ & & J^1 & & \end{array}$$

where the dotted arrow, f^1, exists by the universal property of J^1. Note that $f^1 \circ d_I^0 = f^1 \circ \alpha^0 \circ p^0 = \widehat{d_J^0 \circ f^0} \circ p^0 = d_J^0 \circ f^0$.

To continue inductively, suppose that we have constructed $f^0, f^1, \ldots, f^r$, such that $f^i : I^i \to J^i$, $0 \le i \le r$ and $f^i \circ d_I^{i-1} = d_J^{i-1} \circ f^{i-1}$, for $1 \le i \le r$.

Consider the maps $p^r : I^r \to \operatorname{coker} d_I^{r-1}$. As before, since $\operatorname{im} d_I^{r-1} = \ker d_I^r$, by third isomorphism theorem we again have that $\operatorname{coker} d_I^{r-1} \cong \operatorname{im} d_I^r$ and hence we have an injective map $\alpha^r : \operatorname{coker} d_I^{r-1} \to I^{r+1}$. Also note that $\alpha^r \circ p^r = d_I^r$. As before we also note that $(d_J^r \circ f^r) \circ d_I^{r-1} = d_J^r \circ d_J^{r-1} \circ f^{r-1} = 0$. Therefore, $d_J^r \circ f^r$ factors through $\operatorname{coker} d_I^{r-1}$, that is $d_J^r \circ f^r = \widehat{d_J^r \circ f^r} \circ p^r : I^r \to J^{r+1}$. Thus we have a commutative diagram with an exact top row:

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{coker} d_I^{r-1} & \xrightarrow{\alpha^r} & I^{r+1} \\ & & {\scriptstyle \widehat{d_J^r \circ f^r}}\downarrow & \swarrow {\scriptstyle f^{r+1}} & \\ & & J^{r+1} & & \end{array},$$

where the dotted arrow f^{r+1} exists because of the universal property of the injective module J^{r+1}. Furthermore, $f^{r+1} \circ d_I^r = f^{r+1} \circ \alpha^r \circ p^r = \widehat{d_J^r \circ f^r} \circ p^r = d_J^r \circ f^r$.

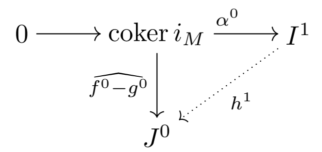
This proves that there exists an extension of f to a cochain homomorphism $f^* : I^* \to J^*$.

Thus the only thing which remains to prove is that any two such extensions are chain homotopic. Suppose $f^*, g^* : I^* \to J^*$ be two cochain homomorphisms which extend f. We shall construct a collection of maps $h^i : I^i \to J^{i-1}$ such that $f^i - g^i = h^{i+1} \circ d_I^i + d_J^{i-1} \circ h^i$. First define $h^i = 0$ for $i \leq 0$. So we start with defining h^1. Note that h^1 should be such that $f^0 - g^0 = h^1 \circ d_I^0$.

We observe that $(f^0 - g^0) \circ i_M = i_N \circ (f - f) = 0$. Therefore $f^0 - g^0$ factors as $f^0 - g^0 = \widehat{f^0 - g^0} \circ p^0$:

$$I^0 \xrightarrow{p^0} \operatorname{coker} i_M \xrightarrow{\widehat{f^0 - g^0}} J^0.$$

As before, we also have the inclusion map $\alpha^0 : \operatorname{coker} i_M \to I^1$. Thus we have a diagram with exact top row:



where the dotted arrow h^1 exists because of the universal property of J^0.

Now we shall define h^i by induction. Suppose we have defined $h^1, \ldots, h^r$, such that $f^i - g^i = h^{i+1} \circ d_I^i + d_J^{i-1} \circ h^i$ for $0 \leq i \leq r - 1$.

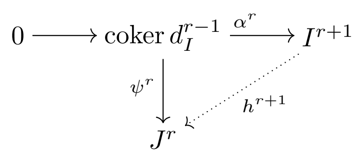
Define $\varphi^r = f^r - g^r - d_J^{r-1} \circ h^r$. We want to define an h^{r+1} such that $\varphi^r = h^{r+1} \circ d_I^r$. Now,

$$\begin{aligned}
\varphi^r \circ d_I^{r-1} &= f^r \circ d_I^{r-1} - g^r \circ d_I^{r-1} - d_J^{r-1} \circ h^r \circ d_I^{r-1} \\
&= d_J^{r-1} \circ f^{r-1} - d_J^{r-1} \circ g^{r-1} - d_J^{r-1} \circ h^r \circ d_I^{r-1} \\
&= d_J^{r-1} \circ (f^{r-1} - g^{r-1} - h^r \circ d_I^{r-1}) \\
&= d_J^{r-1} \circ (d_J^{r-2} \circ h^{r-1}) \text{ (by induction)} \\
&= 0.
\end{aligned}$$

Therefore, φ^r factors as

$$I^r \xrightarrow[p^r]{} \operatorname{coker} d_I^{r-1} \xrightarrow[\psi^r]{} J^r, \qquad \varphi^r = \psi^r \circ p^r.$$

This allows us to draw the following diagram in which the top row is exact:



where h^{r+1} exists by the universal property of the injective module J^r. Now one checks that

$$h^{r+1} \circ d_I^r = h^{r+1} \circ \alpha^r \circ p^r = \psi^r \circ p^r = \varphi^r$$

as we needed. □

Proposition 2. *Suppose I^* and J^* are two injective resolutions of an R module M. Then there exists cochain maps $f^* : I^* \to J^*$ and $g^* : J^* \to I^*$ such that $g^* \circ f^*$ is homotopic to $\mathrm{id}^*_{I^*} : I^* \to I^*$ and $f^* \circ g^*$ is homotopic to $\mathrm{id}^*_{J^*} : J^* \to J^*$.*

Proof. Use the above propositions to prove that extensions of the identity map in either direction will give f^* and g^* respectively. □

Exercise 3. Complete the proof.

3. Ext

Suppose M and N are R modules. Consider an injective resolution I^* of N. Applying $\hom_R(M, \_\_)$ to this complex we get the complex

$$\hom(M, I^*) := 0 \xrightarrow{\delta^{-1}} \hom(M, I^0) \xrightarrow{\delta^0} \hom(M, I^1) \xrightarrow{\delta^1} \cdots$$

Definition 4. $\mathrm{Ext}^i_R(M, N)$ is defined to be

$$H^i(\hom(M, I^*)) = \frac{\ker \delta^i}{\mathrm{im}\, \delta^{i-1}}.$$

Exercise 5. Prove that $\mathrm{Ext}^0_R(M, N) \cong \hom_R(M, N)$.

Lemma 6. *Suppose A^* and B^* are two cochain complexes and $f^* : A^* \to B^*$ and $g^* : B^* \to A^*$ are cochain homomorphisms such that $g^* \circ f^*$ and $f^* \circ g^*$ are homotopic to identity. Then f^* and g^* induce isomorphisms between $H^*(A^*)$ and $H^*(B^*)$ and the induced maps are inverses of each other.*

Proof. This I assumed you know. Otherwise, check any standard book on homological algebra. □

Proposition 7. *The definition of* Ext *is independent of the choice of injective resolution chosen.*

Proof. This I'll leave as a bunch of exercises. Suppose I^* and J^* are two injective resolutions of N. Using proposition 2 we find f^* and g^* as in the above lemma. Applying $\hom(M, \_)$ to everything, conclude that $\circ f^*$ and $\circ g^*$ satisfy the hypothesis if the above lemma for $A^* = \hom(M, I^*)$ and $B^* = \hom(M, J^*)$. This will immediately imply that the cohomologies of these complexes are isomorphic, and hence $\mathrm{Ext}^*(M, N)$ are well defined upto isomorphism. □