The category of modules have enough injectives

Fix R a commutative ring with 1 and let M be an R-module. We want to find an injective R-module $I(M)$ and an injective map $e : M \to I(M)$.

1. Construction of $I(M)$

We saw how to construct injective objects in the category of R-$\mathcal{M}\mathfrak{od}$. We first note that if we take the divisible group $\mathbb{Q}$ we may not get something interesting. The problem happens when R is, say for example, $\mathbb{Z}/p\mathbb{Z}$. In this case $H(\mathbb{Q}) = \hom_{\mathcal{A}\mathfrak{belian}\mathcal{G}\mathfrak{roups}}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) = 0$ and hence this does not give anything useful. However if we take the divisible group to be $\mathbb{Q}/\mathbb{Z}$ things are much better.

Lemma 1.

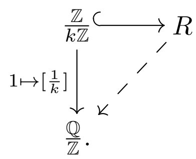
$$I_0 := H\left(\frac{\mathbb{Q}}{\mathbb{Z}}\right) = \hom_{\mathcal{A}\mathfrak{belian}\mathcal{G}\mathfrak{roups}}\left(R, \frac{\mathbb{Q}}{\mathbb{Z}}\right) \neq 0$$

for all rings R.

For the proof we use the fact that $\mathbb{Q}/\mathbb{Z}$ is divisible.

Proof. Consider the group homomorphism $\gamma : Z \to R$ given by $1 \mapsto 1$. There are two cases: either γ is injective, or it has a non-trivial kernel $K \cong k\mathbb{Z}$, $k \geq 2$. Note that k cannot be 1 as $\gamma(1) = 1$ and hence γ cannot be the zero map.

In case γ is not injective, we have an injective map $\mathbb{Z}/k\mathbb{Z} \hookrightarrow R$. Now consider the diagram



Divisibility of $\mathbb{Q}/\mathbb{Z}$ guarantees the existence of the dotted arrow. Therefore, we have a nontrivial (why?) map from $R \to \mathbb{Q}/\mathbb{Z}$.

In case γ is injective, consider any non-trivial map $\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ (for example, $1 \mapsto [\frac{1}{2}]$) and then it extends to R by the same argument as in the previous case.

Existence of a nontrivial map in both the cases imply that $I_0 \neq 0$. ☐

Definition 2. For M an object in R-$\mathcal{M}\mathfrak{od}$, define an indexing set $\mathscr{A} := \hom_{R\text{-}\mathcal{M}\mathfrak{od}}(M, I_0)$. Define

$$I(M) := \prod_{\mathscr{A}} I_0.$$

For sake of making things clear, for $\alpha \in \mathscr{A}$ and $x \in I(M)$, let the α-th component of x be denoted by x^α.

Corollary 3. *Since I_0 is defined as $H(\mathbb{Q}/\mathbb{Z})$, it is an injective R-module. Hence $I(M)$, being a product of injective R-modules, is an injective R-module too.*

2. M embeds into $I(M)$

Define $e_M : M \to I(M) = \prod_{\mathscr{A}} I_0$ by $(e_M(m))^\alpha = \alpha(m)$ where $\alpha \in \mathscr{A} = \hom_{R\text{-}\mathcal{M}\mathfrak{od}}(M, I_0)$.

Exercise 4. Check that $e_M : M \to I(M)$ is an R-module homomorphism.

Proposition 5. *e_M is injective.*

Proof. We have to show that $e_M(m) = 0$ implies $m = 0$. But $e_M(m) = 0$ just means $f(m) = 0$ for all $f \in \hom_{R\text{-}\mathcal{M}\mathfrak{od}}(M, I_0)$. We now prove by contradiction.

Suppose $m \neq 0$. We construct an $f : M \to I_0$, an R-module homomorphism such that $f(m) \neq 0$. Note that such an f belongs to $\hom_{R\text{-}\mathcal{M}\mathfrak{od}}(M, I_0) = \hom_{R\text{-}\mathcal{M}\mathfrak{od}}(M, \hom_{\mathcal{A}\mathfrak{belian}\mathcal{G}\mathfrak{roups}}(R, \mathbb{Q}/\mathbb{Z})) \cong \hom_{\mathcal{A}\mathfrak{belian}\mathcal{G}\mathfrak{roups}}(M, \mathbb{Q}/\mathbb{Z})$.

Exercise 6. Check that if we construct a group homomorphism $g : M \to \mathbb{Q}/\mathbb{Z}$ belonging to the last group above satisfying $g(m) \neq 0$, the corresponding map f belonging to $\hom_{R\text{-}\mathcal{M}\mathfrak{od}}(M, I_0)$ satisfies $f(m) \neq 0$.

To construct such a g look at the abelian group homomorphism $\mathbb{Z} \to M$ given by $1 \mapsto m$. *Check that* the arguments in lemma 1 gives such a g. $\square$