Adjoint functors and injective objects

For the next two lectures, let R be a commutative ring with 1. All the proofs can be suitably modified to cater to the case when R is just an associative ring with 1 and we take all modules. However we shall stick to the commutative case, as that is all we need.

Let R-$\mathfrak{Mod}$ be the category of R-modules.

1. Adjoint functors

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories and suppose we have two functors

$$L : \mathcal{A} \longrightarrow \mathcal{B},$$
$$\text{and } R : \mathcal{B} \longrightarrow \mathcal{A}.$$

Suppose that for any object A in $\mathcal{A}$ and any object B in $\mathcal{B}$, there is a natural isomorphism

$$\tau : \hom_{\mathcal{B}}(LA, B) \xrightarrow{\sim} \hom_{\mathcal{A}}(A, RB).$$

Then L is said to be the *left adjoint* of R and R is said to be the *right adjoint* of L.

Now we give an example of adjoint functors which we shall use soon.

2. Right adjoint of the forgetful functor from R-$\mathfrak{Mod}$ to $\mathfrak{AbelianGroups}$

Consider the functor $\mathit{ff} : R\text{-}\mathfrak{Mod} \to \mathfrak{AbelianGroups}$, where $\mathit{ff}(M)$ is nothing but the underlying commutative group defining M and $\mathit{ff}(\alpha : M \to N)$ is nothing but the group homomorphism underlying the module homomorphism.

Now we want to construct a right adjoint to the functor ff. Clearly it should be a functor from $\mathfrak{AbelianGroups}$ to R-$\mathfrak{Mod}$. Start with an abelian group A. A simple way to construct an R-module is the following.

Consider $H(A) := \hom_{\mathfrak{AbelianGroups}}(R, A)$. I claim that it is an R-module. Of course, $H(A)$ is an abelian group. To see that it is a module consider $r \in R$ and $f \in H(A)$. That is f is a group homomorphism $f : R \to A$. Define $rf : R \to A$ to be the function $(rf)(s) = f(rs)$.

Exercise 1. Check that this definition of a scalar multiplication makes $H(M)$ into an R-module.

As one might guess, now we'll claim that H is a right adjoint to ff. We need to construct a map

$$\tau : \hom_{\mathfrak{AbelianGroups}}(\mathit{ff}(M), A) \longrightarrow \hom_{R\text{-}\mathfrak{Mod}}(M, H(A)).$$

Let $\alpha : \mathit{ff}(M) \to A$ be a map of abelian groups. $\tau(\alpha)$ will be a map from M to $H(A)$. That is $(\tau(\alpha))(m)$ will be in $H(A)$ that it is a group homomorphism $R \to A$. Define

$$((\tau(\alpha))(m))(r) = \alpha(rm).$$

Exercise 2. Check that $\tau(\alpha)$ is an R-module homomorphism and hence τ does give a map as promised.

Exercise 3. Construct a map

$$\mu : \hom_{R\text{-}\mathfrak{Mod}}(M, H(A)) \longrightarrow \hom_{\mathfrak{AbelianGroups}}(\mathit{ff}(M), A).$$

where, for $g \in \hom_{R\text{-}\mathfrak{Mod}}(M, H(A))$, $\mu(g)$ is the map $\mu(g)(m) = (g(m))(1)$. Prove that μ is the inverse of τ proving that τ is an isomorphism.

Exercise 4. Convince yourself that μ and τ are natural isomorphisms.

Thus we proved that H is a right adjoint to ff.

3. Injectives and right adjoints

Recall that an object I in a category R-$\mathcal{M}\mathfrak{od}$ is said to be *injective* if for any monomorphism $i : M \to N$, and a morphism $\alpha : M \to I$, there exists a morphism (not necessarily unique) $\beta : N \to I$ such that the following diagram commutes.

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{i} & N \\ & & {\scriptstyle\alpha}\downarrow & \swarrow_{\exists\beta} & \\ & & I & & \end{array}$$

Proposition 5. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are categories of modules over some rings. Suppose $R : \mathcal{A} \to \mathcal{B}$ and $L : \mathcal{B} \to \mathcal{A}$ be functors such that R is a right adjoint to L. Suppose I is an injective object in $\mathcal{A}$. Then $R(I)$ is an injective object in $\mathcal{B}$.*

Proof. First note that, I is injective if and only if for any monomorphism $i : M \to N$, $\hom_{\mathcal{A}}(N, I) \xrightarrow{\circ i} \hom_{\mathcal{A}}(M, I)$ is surjective. Thus we have to check that for any monomorphism $P \to Q$ in $\mathcal{B}$, $\hom_{\mathcal{B}}(Q, R(I)) \to \hom_{\mathcal{B}}(P, R(I))$ is surjective. This immediately follows from the commutativity (check it!) of the following diagram

$$\begin{array}{ccc} \hom_{\mathcal{B}}(Q, R(I)) & \longrightarrow & \hom_{\mathcal{B}}(P, R(I)) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \hom_{\mathcal{A}}(L(Q), I) & \twoheadrightarrow & \hom_{\mathcal{A}}(L(P), I). \end{array}$$

□

Corollary 6. *For every divisible abelian group D, $\hom_{\mathcal{A}\mathfrak{belian}\mathcal{G}\mathfrak{roups}}(R, D)$ is an injective R-module.*