Some more homological algebra

There are lot of references for these results. I'll just give the references in "Dummit and Foote", *Algebra* and "Weibel" *Introduction to Homological Algebra.*

1. Snake Lemma

Theorem 1. *Consider the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{i_M} & M_2 & \xrightarrow{p_M} & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & M_1 & \xrightarrow{i_M} & M_2 & \xrightarrow{p_M} & M_3 & \longrightarrow & 0. \end{array}$$

Then we have an exact sequence

$$0 \longrightarrow \ker f_1 \longrightarrow \ker f_2 \longrightarrow \ker f_3 \longrightarrow \operatorname{coker} f_1 \longrightarrow \operatorname{coker} f_2 \longrightarrow \operatorname{coker} f_3 \longrightarrow 0.$$

Proof. Weibel : Snake Lemma 1.3.2 (page 11)
Dummit and Foote : Exercises 2-3 (pages 791-2). □

2. Injective and Projective resolutions

Throughout this lecture, let R be a commutative ring with 1. We are interested in studying modules over R and how they behave under tensor products and (internal) Homs.

Definition 2. An R-module I is said to be *injective* if for any inclusion of R-modules $M \subset N$, any homomorphism $M \to I$ can be extended to a homomorphism from $N \to I$.

To give some examples, we introduce the notion of divisibility.

Definition 3. An abelian group (a $\mathbb{Z}$-module) D is said to be divisible if for any object $x \in D$ and any positive integer n, there exists an element $y \in D$ such that $ny = \underbrace{y + \cdots + y}_{n\text{-times}} = x$.

Proposition 4. *Any abelian group I is injective if and only if it is divisible.*

Proof. The proof is quite clear in Bredon. The main steps are as follows.

1. To check injective objects are divisible. Let I be an injective abelian group. Let $x \in I$ and $n \in \mathbb{Z}$, $n > 0$. We want to show that there exists y such that $ny = x$.
   - (a) Consider the maps $0 \to n\mathbb{Z} \to Z$ and look at the map from $n\mathbb{Z} \to I$ given by $n \mapsto x$.
   - (b) This map extends to a map from $\mathbb{Z} \to I$. Then the image of 1 is y.

(2) To check that divisible objects are injective. Suppose D is a divisible group. Let $0 \to H \to G$ be an exact sequence. Suppose we are given a map $g : H \to D$. We have to show that it extends to a map from $G \to D$.

(a) Let $\mathscr{A} = \{(H', g') \mid H \subset H' \subset G, \text{ and } g' : H' \to D \text{ extends } g\}$. Note that $(H, g) \in \mathscr{A}$ implies that $\mathscr{A}$ is non-empty. Moreover, one shows that every chain has a maximal element and hence *Zorn's lemma* will imply that there exists a maximal element, say (G', h). If $G' = G$, we are done. If $G' \neq G$, let $x \in G \setminus G'$.

(b) If $nx \notin G'$ for all $n > 0$, then define

$$h' : G'' = \{g' + kx \mid g' \in G, k \in Z\} \to D$$

by $h'(g' + kx) = h(g')$. Then one can sees $(G'', h') \in \mathscr{A}$ contradicting maximality of (G', h).

(c) If $rx \in G'$ for $r > 1$ being the minimal r satisfying $rx \in G'$, Let $h(rx) = d$. Let d' be the element such that $rd' = d$. Then define $G'' = G + \mathbb{Z}x$ and $h'(g' + kx) = h(g') + kd'$. Then check that $(G'', h') \in \mathscr{A}$ again contradicting the maximality of (G', h).

This completes the proof.

□

Proposition 5. *Any abelian group can be embedded in a divisible group.*

Proof. Consider an abelian group A and let $\{a_\alpha\}_\alpha$ be its generators. Then we have a surjection

$$\upsilon : \bigoplus_\alpha \mathbb{Z} \to A$$

where the $1 \in \mathbb{Z}_\alpha$ (the α-th component of the direct sum above) goes to a_α. Let $K = \ker \upsilon$. This means $A \cong (\bigoplus_\alpha \mathbb{Z})/K$. Let

$$D(A) = \frac{\bigoplus_\alpha \mathbb{Q}}{K}.$$

Then $A = (\bigoplus_\alpha \mathbb{Z})/K \hookrightarrow D(A)$ is an injection and it is clear that $D(A)$ is a divisible group. □