Towards Universal Coefficient theorems

1. Tensor products and Homs

This is a nice place to introduce adjoint functors. Let us start with an example.

Definition 1. Let A and B be abelian groups. A bilinear map $\varphi : A \times B \to C$ is a map to another abelian group C such that

- $\varphi(a + a', b) = \varphi(a, b) + \varphi(a', b)$, and
- $\varphi(a, b + b') = \varphi(a, b) + \varphi(a, b')$.

The tensor product of A and B is a bilinear map $\psi : A \times B \to T$ such that for any other bilenar map $\varphi : A \times B \to C$ as above, we have a unique map $\alpha : T \to C$ making the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\psi} & T \\ & \overset{\varphi}{\searrow} & \downarrow \alpha \\ & & C \end{array}$$

commutative.

I'll assume that you know why tensor products exist. Consider a short exact sequence of abelian groups

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0.$$

Then for any abelian group B we have an exact sequence

$$A' \otimes B \xrightarrow{f \otimes \mathrm{id}_B} A \otimes B \xrightarrow{g \otimes \mathrm{id}_B} A'' \otimes B \longrightarrow 0$$

Now recall that giving a bilinear map $A \times B \to C$ is equivalent to giving a group homomorphism from $A \to \hom(B, C)$.

$$\varphi : A \times B \to C \longmapsto \alpha : a \mapsto \varphi(a, \_) \in \hom(A, \hom(B, C))$$

$$a, b \mapsto \alpha(a)b \in \hom(A \times B, C) \longleftarrow\!\!\dashv \alpha : A \to \hom(B, C)$$

Also by the definition of tensor product, giving a bilinear map $A \times B \to C$ is equivalent to giving a group homomorphism $A \otimes B \to C$. In particular, we get that

$$\hom(A \otimes B, C) \cong \hom(A, \hom(B, C)).$$

Consider the covariant functors

$$F = \_ \otimes B : \mathfrak{AbelianGroups} \to \mathfrak{AbelianGroups}$$
$$G = \hom(B, \_) : \mathfrak{AbelianGroups} \to \mathfrak{AbelianGroups}.$$

Then rewriting the above equation we get that

$$\hom(F(A), C) \cong \hom(A, G(C))$$

and this isomorphism is "functorial" in the sense it "behaves well" with morphisms. (Draw the relevant diagrams on the board.)

We shall give a name to this phenomenon

Definition 2. Suppose $\mathcal{A}$ and $\mathcal{B}$ be two categories. And suppose $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be two functors. Then F is said to be the *left adjoint* of G if for all objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there is a functorial isomorphism

$$\hom_{\mathcal{B}}(F(A), B) = \hom_{\mathcal{A}}(A, G(B)).$$

Here as before functorial means that for morphisms $A \to A'$ and $B \to B'$ the induced (following) diagrams of maps commute.

$$\begin{array}{ccc} \hom_{\mathcal{B}}(F(A'), B) & \longrightarrow & \hom_{\mathcal{A}}(A', G(B)) \\ \downarrow & & \downarrow \\ \hom_{\mathcal{B}}(F(A), B) & \longrightarrow & \hom_{\mathcal{A}}(A, G(B)) \end{array}$$

$$\begin{array}{ccc} \hom_{\mathcal{B}}(F(A), B) & \longrightarrow & \hom_{\mathcal{A}}(A, G(B)) \\ \downarrow & & \downarrow \\ \hom_{\mathcal{B}}(F(A), B') & \longrightarrow & \hom_{\mathcal{A}}(A, G(B')). \end{array}$$

In this case G is said to be the right adjoint of F.

Remark 3. Here the *left* in left adjoint stands for the fact that F appears in the left inside the homomorphism. Similarly for right.

Thus what we saw above was that $\_ \otimes B$ is the left adjoint to $\hom(B, \_)$ for all abelian groups B.

Now we list some facts about tensor products. You can do them as exercises.

(1) $\mathbb{Z} \otimes B \cong B$.

(2) Tensor product $\_ \otimes B$ is right exact.[1] Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\bmod 2} \frac{\mathbb{Z}}{2\mathbb{Z}} \longrightarrow 0$$

and tensor it with $\mathbb{Z}/2\mathbb{Z}$ to show that the result sequence is not exact on the left.

(3) $\hom(\_, B)$ is left exact. Give an example where it fails to be right exact.

(4) $(\oplus_\alpha A_\alpha) \otimes B \cong \oplus_\alpha (A_\alpha \otimes B)$.

(5) Suppose

$$0 \longrightarrow A' \xrightarrow{f} A \underset{s}{\overset{g}{\rightleftarrows}} A'' \longrightarrow 0$$

be a short exact sequence which is split, that is, has a map s as above such that $g \circ s = \mathrm{id}_{A''}$. Show that for any B, $\hom(\_, B)$ applied to this sequence is exact.

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[1] A functor F is right exact if for every short exact sequence $0 \to A' \to A \to A'' \to 0$, $F(A') \to F(A) \to F(A'') \to 0$. A functor G is left exact if for every short exact sequence $0 \to B' \to B \to B'' \to 0$ $0 \to G(A') \to G(A'') \to G(A'')$ is exact. A functor is said to be exact if it is both left and right exact.

2. Operations on chains

Let $C = (C_*, \partial)$ be a chain complex. Consider an abelian group G. Then $C \otimes G := (D_*, \partial \otimes \mathrm{id}_G)$ where $D_i = C_i \otimes G$ is also a chain complex, since $(\partial \otimes \mathrm{id}_G) \circ (\partial \otimes \mathrm{id}_G) \sum_i a_i \otimes g_i = \sum_i \partial(\partial(a_i)) \otimes g_i = \sum_i 0 \otimes g_i = 0$. On the other hand, as we did before, $\hom(C, G) := (E^*, \delta)$ where $E^i = \hom(C_i, G)$ and δ is defined by $\alpha \in \hom(C_i, G) \mapsto \alpha \circ \partial : \hom(C_{i+1}, G)$. Here the index of E^i is written as a superscript to emphasize the fact that here the (coboundary) map goes from $E^i \to E^{i+1}$. It is again easy to see that $\delta \circ \delta = 0$

Now consider the category $\mathfrak{ChainComplex}$ of chain complexes of abelian groups. In the definition of chain complexes the boundary maps go from $E_i \to E_{i-1}$. The category of cochain complexes $\mathfrak{CochainComplex}$ consists of complexes where the maps go from $E^i \to E^{i+1}$. Now we claim that tensor induces a map $\_ \otimes G : \mathfrak{ChainComplex} \to \mathfrak{ChainComplex}$ and $\hom(\_, G) : \mathfrak{ChainComplex} \to \mathfrak{CochainComplex}$.

We already know where the objects go. Now let $f : (C_*, \partial) \to (D_*, \partial_D)$ Consider $\_ \otimes G$ first. We would get a diagram like

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial\otimes\mathrm{id}_G} & C_{i-1} \otimes G & \xrightarrow{\partial\otimes\mathrm{id}_G} & C_i \otimes G & \xrightarrow{\partial\otimes\mathrm{id}_G} & C_{i+1} \otimes G & \xrightarrow{\partial\otimes\mathrm{id}_G} & \cdots \\
 & & \downarrow{\scriptstyle f\otimes\mathrm{id}_G} & & \downarrow{\scriptstyle f\otimes\mathrm{id}_G} & & \downarrow{\scriptstyle f\otimes\mathrm{id}_G} & & \\
\cdots & \xrightarrow[\partial_D\otimes\mathrm{id}_G]{} & D_{i-1} \otimes G & \xrightarrow[\partial_D\otimes\mathrm{id}_G]{} & D_i \otimes G & \xrightarrow[\partial_D\otimes\mathrm{id}_G]{} & D_{i+1} \otimes G & \xrightarrow[\partial_D\otimes\mathrm{id}_G]{} & \cdots .
\end{array}$$

Check that all the squares in the above diagram are commutative. Thus the functor $\_ \otimes G$ is covariant.

$\hom(\_, G)$ is contravariant. For the same f as above, $f^* := \hom(f, G) : \hom(D, G) \to \hom(C, G)$ where $f^*(\beta) = \beta \circ f \in \hom(C_i, G)$ for all $\beta \in \hom(D_i, G)$.

Exercise 4. Draw a diagram like above for the case of $\hom(,G)$ and prove that all the square commute and hence $\hom(f, G)$ is a morphism of cochains.

3. Singular homology and cohomology with coefficients in a group

Consider a chain complex (C_*, ∂). We define

$$H_i(C_*, \partial) = \frac{\ker(\partial : C_i \to C_{i-1})}{\mathrm{im}(\partial : C_{i+1} \to C_i)};$$

and for a cochain complex (E^*, δ), one defines the hypercohomology

$$H^i(E^*, \delta) = \frac{\ker(\delta : E^{i-1} \to E^i)}{\mathrm{im}(\delta : E^i \to E^{i+1})}.$$

Example 5 (Singular homology). For a topological space X, consider the chain complex (C_*, ∂) where $C_i = \Delta_i(X)$ and ∂ are given by linear combination of the face maps. In this case $H_*(C_*, \partial)$ is nothing by the homology groups. The homology groups with coefficients in G are nothing but $H_*(C \otimes G)$.

As before, define $\Delta^i(X; G)$ as before to be to be group $\hom(\Delta_i(X), G)$.

Definition 6. Define the singular cohomology $H^*(X; G)$ of X with coefficients in G to be the hypercohomology groups $H^*(\hom((C_*, \partial), G))$ where C is as in the example.