Singular cohomology

1. Some unanswered questions from last week

We list the questions which we promised to answer last week. We'll also list the ones which we do not want to spend time on.

(1) Stokes' theorem for manifolds with corners. Suppose ω is a compactly supported $(p-1)$-form on Δ_p. Then,
$$\int_{\Delta_p} d\omega = \int_{\partial\Delta_p} \omega.$$
I won't go into the details. But one can "approximate" Δ_p as a union of an increasing sequence of manifolds with boundary. Then one has to check that all the limits make sense and they give the desired result.

(2) Orientability and existence of a nowhere vanishing top form. This will be done in the next section.

2. Orientability revisited

Consider an oriented manifold M of dimension m. Consider $\mathscr{A}$ be an oriented atlas on M. First we shall construct a nowhere vanishing m-form on M. For this suppose $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M such that $(U_\alpha,\, \varphi_{U_\alpha}) \in \mathscr{A}$. Suppose $x_1^\alpha, \ldots, x_m^\alpha$ be the local coordinates corresponding to $(U_\alpha,\, \varphi_{U_\alpha})$. Then $\omega_\alpha = dx_1^\alpha \wedge \cdots \wedge dx_m^\alpha$ is a nowhere vanishing m form on U_α.

Now let f_β be a partition of unity of M subordinate to U_α. Suppose $\alpha(\beta) \in A$ is the index such that $\operatorname{supp} f_\beta \subset U_\alpha$. Define
$$\omega = \sum_\beta f_\beta \omega_{\alpha(\beta)}.$$
Check that ω is a nowhere vanishing m-form on M.

The only thing left to check is that a nowhere vanishing m-form ω on M defines an orientation on M. For this consider,
$$\mathscr{B} = \{(U_\alpha,\, \varphi_{U_\alpha}) \mid \forall p \in U_\alpha,\ \omega_p = a(dx_1^\alpha \wedge \cdots \wedge dx_m^\alpha)_p \text{ with } a > 0\}$$
where $x_1^\alpha, \ldots, x_m^\alpha$ are the local coordinates for $(U_\alpha,\, \varphi_{U_\alpha})$.

Exercise 1. $\mathscr{B}$ is an oriented atlas on M.

Now we can recall the definition of an orientation preserving map.

Definition 2. A C^∞ map $F : M \to N$ of oriented manifolds, with orientations being given by the forms ω_M and ω_N respectively, is said to be *orientation preserving*, if $F^*\omega_N$ is nowhere vanishing and at each point $p \in M$, $(\omega_M)_p = a(F^*\omega_N)_p$ with $a > 0$.

3. A map from deRham to singular cohomology

First a definition. Let $\Delta^p(M;\mathbb{R}) = \hom(\Delta_p(M), \mathbb{R})$ where the hom denotes homomorphism of groups. Note that $\Delta^p(M;\mathbb{R})$ is an abelian group.

First we construct a map $\Psi : \Omega^p(M) \to \Delta^p(M;\mathbb{R})$. Let ω be a p-form. Define $\Psi(\omega)$ to be the group homomorphism defined by
$$\Psi(\omega)(c) = \int_c \omega.$$

This map defines a group homomorphism as required. Now recall that we have two chain complexes

$$\cdots \xrightarrow{d} \Omega^{p-1}(M) \xrightarrow{d} \Omega^{p}(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \cdots$$

and

$$\cdots \xleftarrow{\partial} \Delta_{p-1}M \xleftarrow{\partial} \Delta_{p}(M) \xleftarrow{\partial} \Delta_{p+1}(M) \xleftarrow{\partial} \cdots .$$

However in the second sequence the maps go in the wrong direction. But we can consider the dual map

$$\cdots \xrightarrow{\delta} \Delta^{p-1}(M;\mathbb{R}) \xrightarrow{\delta} \Delta^{p}(M;\mathbb{R}) \xrightarrow{\delta} \Delta^{p+1}(M;\mathbb{R}) \xrightarrow{\delta} \cdots .$$

where $\delta : \Delta^p(M;\mathbb{R}) \to \Delta^{p+1}(M;\mathbb{R})$ is defined by

$$(\delta f)c = f(\partial c)$$

for any $f \in \Delta^p(M;\mathbb{R})$ and for any $c \in \Delta_{p+1}(M)$.

Combining the maps Ψ, δ and d, we get a diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & \Omega^{p-1}(M) & \xrightarrow{d} & \Omega^{p}(M) & \xrightarrow{d} & \Omega^{p+1}(M) & \xrightarrow{d} & \cdots \\ & & \downarrow \Psi & & \downarrow \Psi & & \downarrow \Psi & & \\ \cdots & \xrightarrow{\delta} & \Delta^{p-1}(M;\mathbb{R}) & \xrightarrow{\delta} & \Delta^{p}(M;\mathbb{R}) & \xrightarrow{\delta} & \Delta^{p+1}(M;\mathbb{R}) & \xrightarrow{\delta} & \cdots \end{array} \tag{1}$$

Now we check that the diagram is commutative. For that we need $\Psi \circ d = \delta \circ \Psi$. Consider $\omega \in \Omega^{p-1}(M)$. $\Psi(d\omega) \in \Delta^p(M;\mathbb{R})$. Now for any $\sigma \in \Delta_p(M)$,

$$\Psi(d\omega)(\sigma) = \int_\sigma d\omega = \int_{\Delta_p} \sigma^* d\omega = \int_{\Delta_p} d(\sigma^*\omega)$$

by Stokes' theorem

$$= \int_{\partial\Delta_p} \sigma^*\omega$$

and since $\partial\Delta_p = \sum_{i=1}^{p+1}(-1)^i F_i$,

$$\begin{aligned} &= \sum_{i=1}^{p+1}(-1)^i \int_{F_i} \sigma^*\omega = \sum_{i=1}^{p+1}(-1)^i \int_{\Delta_{p-1}} {F_i}^*\sigma^*\omega \\ &= \sum_{i=1}^{p+1}(-1)^i \int_{\sigma\circ F_i} \omega = \int_{\partial\sigma} \omega = \Psi(\omega)(\partial\sigma) \\ &= (\delta\Psi(\omega))(\sigma) \end{aligned}$$

proving the claim. Thus the diagram (1) is *commutative.*

Thus if we define

$$H^p(M;\mathbb{R}) = \frac{\ker(\delta : \Delta^p(M;\mathbb{R}) \to \Delta^{p+1}(M;\mathbb{R}))}{\operatorname{im}(\delta : \Delta^{p-1}(M;\mathbb{R}) \to \Delta^{p}(M;\mathbb{R}))}$$

we get maps $\Psi^* : H^p(M;\mathbb{R}) \to H^p_\Omega(M)$.

Later on we shall prove that Ψ^* is an isomorphism for all manifolds M.

The groups defined above $H^p(M;\mathbb{R})$ are called the *singular cohomology groups.*

4. Some more questions and comments

We end the lecture with some more questions and comments.

(1) We defined $\Delta_p(M)$ as the free group on smooth maps. We can also do so by looking at all continuous maps. That will then make sense for arbitrary topological spaces. Do we get a new definition of singular cohomology in that case?

The answer is no. But I'll prove it eventually. But before that we need to develop some homological algebra which we shall do on the way.

(2) How do we compute any of these groups? Good question. It will turn out that the methods we develop to prove the isomorphism mentioned above, will also give us means to compute.

(3) Finally, we were looking for a relationship between singular homology and deRhan cohomology, but ended up defining singular cohomology. How is singular homology and cohomology related? We shall prove Poincaré duality which will give a relationship in certain cases.

(4) We saw that the wedge product of forms induced a ring structure on the total deRham cohomology. What does this structure correspond to under the above isomorphisms? We shall answer that when we define cup products on singular cohomology.