Relationship between deRham cohomology and singular homology

This is an overview of what is to come later on. First we start by recalling some of the definitions and modifying them to suit our purpose.

1. Concepts from singular homology

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ be the standard basis for $\mathbb{R}^{n+1}$. Define an n-simplex Δ_n to be

$$\Delta_n = \left\{ (x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} x_i \mathbf{e}_i \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1, x_j \geq 0 \; \forall \; 1 \leq j \leq n+1 \right\}.$$

We denote the face generated by $\{\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k}\}$ by

$$[\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k}] = \left\{ \sum_{j=1}^{k} a_j \mathbf{e}_{i_j} \in \mathbb{R}^{n+1} \mid \sum_{j=1}^{k} a_j = 1, a_l \geq 0 \; \forall \; 1 \leq l \leq k \right\}.$$

In particular, $\Delta_n = [\mathbf{e}_1, \dots, \mathbf{e}_{n+1}]$. Define face maps to be inclusions

$$F_i = F_i^p : [\mathbf{e}_1, \dots, \widehat{\mathbf{e}_i}, \dots, \mathbf{e}_{n+1}] \hookrightarrow \Delta_p.$$

A *singular* p simplex of a topological space X is a map $\sigma : \Delta_p \to X$. On the otherhand, we have a manifold, M, and we define a *smooth* p simplex to be a smooth map

$$\sigma : \Delta_p \to M.$$

Here note that Δ_p is not a manifold. What we mean is that σ is a restriction of some smooth map $\tau : U \to M$, where U is an open set containing Δ_p.

Recall that to define homology, one considers the free abelian group generated by p simplices. Elements of that abelian group were called *chains*. We shall be considering *smooth chains*, the elements of the free abelian group generated by smooth p simplices. In particular, a p chain for us will have the form

$$c = \sum_{\sigma} n_\sigma \sigma$$

where σ vary over all smooth simplices, and all but finitely many of the n_σ are non-zero. **From now on every simplex or chain will be assumed to be smooth.**

Now for a simplex σ, define the i-th face to be

$$\sigma^{(i)} = \sigma \circ F_i.$$

We can define the boundary of a simplex to be the $(p-1)$ chain

$$\partial\sigma = \partial_p \sigma = \sum_{i=0}^{p} (-1)^i \sigma^{(i)}.$$

Recall that $\partial \circ \partial = 0$. We shall be proving the following result next week.

> *Fixing an orientation on an oriented n dimensional manifold is equivalent to choosing a nowhere vanishing n form.*

Fix an orientation on Δ_0. Now one orients Δ_p recursively by demanding that the map $F_0 : \Delta_{p-1} \to \partial\Delta_p$ preserves orientation. Here preserving orientation means that pullback of the form defining orientation on $\partial\Delta_p$ differs from the form defining orientation on Δ_{p-1} by a factor of a everywhere positive function.

Now for any p-form ω and a simplex $\sigma : \Delta_p \to M$, one can define

$$\int_\sigma \omega = \int_{\Delta_p} \sigma^* \omega.$$

Note that Δ_p is not a smooth manifold. But one can still define the integral by writing Δ_p as an increasing union of smooth open subsets of $\mathbb{R}^p$. We won't go into the details.

Now for a cycle $c = \sum_\sigma n_\sigma \sigma$, one defines

$$\int_c \omega = \sum_\sigma n_\sigma \int_\sigma \omega.$$