Integration of forms and Stokes' theorem

1. Integration of forms

Definition 1. Let M be a manifold of dimension m and $(U,\, \varphi_U)$ be any coordinate chart, *compatible with the orientation.* $Q \subset U$ is said to be a *cube*, if $\varphi_U(Q)$ is the unit cube, C, in $\mathbb{R}^m$, that is

$$\varphi_U(Q) = C = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m \mid -1 \leq x_i \leq 0, i = 1, \dots, m\}.$$

Note that the notion of a cube is strongly dependent on the choice of the coordinate neighbourhood.

Suppose ω is a form such that $\operatorname{supp}(\omega) \subset Q^\circ$ for some cube Q relative to some coordinate chart $(U,\, \varphi_U)$. Suppose $x_1, \dots, x_m$ are the local coordinates. Then $\omega = f dx_1 \wedge \cdots \wedge dx_m$ for some C^∞ function f.

Definition 2. For such an M, one defines

$$\int_M \omega = \int_{C=[-1,0]^m} (\varphi_U^{-1})^* \omega = \int_{[-1,0]^m} f dx_1 \wedge \cdots \wedge dx_m = \int_0^1 \cdots \int_0^1 f dx_1 \cdots dx_m$$

Lemma 3. *The integral is well defined.*

Proof. Suppose $\operatorname{supp}(\omega) \subset Q'^\circ$ another cube. Suppose the associated coordinate chart is $(V,\, \psi_V)$ with local coordinates $y_1, \dots, y_m$. Let $\omega = g dy_1 \wedge \cdots \wedge dy_m$. Let $\tilde{x}_i = x_i \circ \varphi_U^{-1}$ and $\tilde{y}_i = y_i \circ \psi_V^{-1}$. Now we compute $(\varphi_U^{-1})^*\omega$ in two ways.

Let $\tilde{f} = f \circ \varphi_U^{-1}$ and $\tilde{g} = g \circ \psi_V^{-1}$. Let $A = \psi_V \circ \varphi_U^{-1}$.

(1) First we use $\omega = f dx_1 \wedge \cdots \wedge dx_m$.

$$\begin{aligned}(\varphi_U^{-1})^* f dx_1 \wedge \cdots \wedge dx_m &= (f \circ \varphi_U^{-1}) d(x_1 \circ \varphi_U^{-1}) \wedge \cdots \wedge d(x_m \circ \varphi_U^{-1}) \\ &= \tilde{f} d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_m.\end{aligned}$$

(2) On the other hand, since $\omega = g dy_1 \wedge \cdots \wedge dy_m$,

$$\begin{aligned}(\varphi_U^{-1})^*\omega &= (\psi_V^{-1} \circ A)^* g dy_1 \wedge \cdots \wedge dy_m \\ &= A^* \left((g \circ \psi_V^{-1}) d(y_1 \circ \psi_V^{-1}) \wedge \cdots \wedge d(y_m \circ \psi_V^{-1}) \right) \\ &= A^* \left(\tilde{g} d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m \right) \\ &= (\tilde{g} \circ A) A^*(d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m).\end{aligned}$$

Exercise 4. Show that $A^*(d\tilde{y}_1 \wedge \cdots \wedge d\tilde{y}_m) = (\det JA) d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_m$, where JA is the Jacobian of A.

Therefore we get $(\varphi_U^{-1})^*\omega = (\tilde{g} \circ A)(\det JA) d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_m$.

Putting the two computations together, we conclude that $\tilde{f} = (\tilde{g} \circ A)(\det JA)$.

Now considering ω as $f dx_1 \wedge \cdots \wedge dx_m$, we get

$$\int_M \omega = \int_{[-1,0]^m} \tilde{f} dx_1 \cdots dx_m = \int_{[-1,0]^m} (\tilde{g} \circ A)(\det JA) dx_1 \cdots dx_m.$$

We want this to be equal to

$$\int_{[-1,0]^m} \tilde{g} dy_1 \cdots dy_m.$$

However, since the function inducing the change of coordinates from $(x_1, \ldots, x_m)$ to $(y_1, \ldots, y_m)$ is given by A, we have that

$$\int_{[-1,0]^m} \tilde{g} dy_1 \cdots dy_m = \int_{[-1,0]^m} (\tilde{g} \circ A) |\det JA| dx_1 \cdots dx_m.$$

Since the coordinates are chosen to be compatible with the orientation, $\det JA > 0$ at every point of $\varphi_U(U \cap V)$. Therefore $\int_M \omega = \int_{[-1,0]^m} \tilde{g} dy_1 \cdots dy_m$. This finishes the proof. □

Now let ω be any m-form with compact support on M. Then we define the integral as follows.

Let $\{(U_\alpha, f_\alpha)\}_{\alpha \in I}$ be an partition of unity on M so that $\mathrm{supp}(f_\alpha) \subset (Q_\alpha)^\circ$ for some cube Q_α on M.

Note that $\mathrm{supp}(\omega)$, being compact, will intersect only finitely many of the U_α as is proved in the following exercise.

Exercise 5. Let $\{U_\alpha\}$ be a locally finite covering of a manifold M. Let $K \subset M$ be compact. Then K intersects only finitely many of the U_αs.

Define

$$\int_M \omega = \sum_{\alpha \in I} f_\alpha \omega.$$

To see this is well defined, note that if $\{g_\beta\}_{\beta \in J}$ is any other partition of unity, Then,

$$1 = \sum_{\beta \in J} g_\beta \implies f_\alpha = \sum_{\beta \in J} f_\alpha g_\beta.$$

Also note that $\sum_{\alpha \in I} \int_M f_\alpha g_\beta \omega = \int_M (\sum_{\alpha \in I} f_\alpha) g_\beta \omega = \int_M g_\beta \omega$. Also since K is compact, it intersects finitely many of the open sets in the partition of unity.

Therefore,

$$\sum_{\beta \in J} \int_M g_\beta \omega = \sum_{\beta \in J} \int_M (\sum_{\alpha \in I} f_\alpha g_\beta \omega) = \sum_{\alpha \in I} \int_M (\sum_{\beta \in J} f_\alpha g_\beta \omega) = \sum_{\alpha \in I} \int_M f_\alpha \omega.$$