**Pullback of forms, Integration, Stokes' theorem**

## 1. Properties of pull back of forms

**Proposition 1.** *If $\Phi : M \to N$ and $\Psi : N \to P$ are smooth mappings of $C^\infty$ manifolds, then*

1. $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$;
2. $\Phi^*(\omega \wedge \eta) = \Phi^*(\omega) \wedge \Phi^*(\eta)$;
3. $\Phi^*(f dy_1 \wedge \cdots \wedge dy_p) = (f \circ \Phi) d(y_1 \circ \Phi) \wedge \cdots \wedge d(y_p \circ \Phi)$; *and*
4. $\Phi^*(d\omega) = d(\Phi^*\omega)$.

*Proof.* We shall do them in order.

1. Let $\omega$ be any $p$-form on $P$. Let $X_1, \ldots, X_p$ be a collection of vector fields on $M$. Then,

$$\begin{aligned}(\Psi \circ \Phi)^*\omega(X_1, \ldots, X_p) &= \omega((\Psi \circ \Phi)_* X_1, \ldots, (\Psi \circ \Phi)_* X_p) \\ &= \omega(\Psi_*\Phi_* X_1, \ldots, \Psi_*\Phi_* X_p)) = \Psi^*\omega(\Phi_* X_1, \ldots, \Phi_* X_p) \\ &= (\Phi^* \circ \Psi^*)\omega(X_1, \ldots, X_p).\end{aligned}$$

2. Proved similarly. *Exercise.*
3. Note that $\Phi^*(f) = f \circ \Phi$ and $\Phi^*(df)X = df(\Phi_* X) = (\Phi_* X)f = X(f \circ \Phi) = d(f \circ \Phi)X$ proving that $\Phi^* df = d(f \circ \Phi)$.
   Thus $\Phi^*(f dy_1 \wedge \cdots \wedge dy_p) = \Phi^*(f)\Phi^* dy_1 \wedge \cdots \wedge \Phi^* dy_p = (f \circ \Phi) d(y_1 \circ \Phi) \wedge \cdots \wedge d(y_p \circ \Phi)$.
4. Now the rest is easy. On $(U, \varphi_U)$ with local coordinates $x_1, \ldots, x_m$, suppose $\omega|_U = \sum_{i=1}^m f_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

$$\begin{aligned}(\Phi^*(d\omega))|_U &= \Phi^*(d\omega|_U) = \Phi^* d\left(\sum_{i=1}^m f_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}\right) \\ &= \Phi^* \sum_{i=1}^m df_{i_1,\ldots,i_p} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \\ &= \sum_{i=1}^m \Phi^* df_{i_1,\ldots,i_p} \wedge \Phi^* dx_{i_1} \wedge \cdots \wedge \Phi^* dx_{i_p} \\ &= \sum_{i=1}^m d(f_{i_1,\ldots,i_p} \circ \Phi) \wedge d(x_{i_1} \circ \Phi) \wedge \cdots \wedge d(x_{i_p} \circ \Phi) \\ &= \sum_{i=1}^m d\left((f_{i_1,\ldots,i_p} \circ \Phi) \wedge d(x_{i_1} \circ \Phi) \wedge \cdots \wedge d(x_{i_p} \circ \Phi)\right)\end{aligned}$$

$$\begin{aligned}&= d\left(\sum_{i=1}^m (f_{i_1,\ldots,i_p} \circ \Phi) \wedge d(x_{i_1} \circ \Phi) \wedge \cdots \wedge d(x_{i_p} \circ \Phi)\right) \\ &= d(\Phi^*\omega).\end{aligned}$$

□

Note that the above theorem says that we have a commuative diagram of the following form

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & \Omega^{k-1}(N) & \xrightarrow{d} & \Omega^k(N) & \xrightarrow{d} & \Omega^{k+1}(N) & \xrightarrow{d} & \cdots \\ & & \downarrow{\scriptstyle \Phi^*} & & \downarrow{\scriptstyle \Phi^*} & & \downarrow{\scriptstyle \Phi^*} & & \\ \cdots & \xrightarrow{d} & \Omega^{k-1}(M) & \xrightarrow{d} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(N) & \xrightarrow{d} & \cdots \end{array}$$

and hence by standard homological algebra arguments we get an induced map

$$\frac{\ker d : \Omega^k(N) \to \Omega^{k+1}(N)}{\operatorname{im} d : \Omega^{k-1}(N) \to \Omega^k(N)} \to \frac{\ker d : \Omega^k(M) \to \Omega^{k+1}(M)}{\operatorname{im} d : \Omega^{k-1}(M) \to \Omega^k(M)}$$

define $\Phi^* : H_\Omega^\bullet(N) \to H_\Omega^\bullet(M)$.

$\Phi^*$ is a ring homomorphism $H_\Omega^\bullet(N) \to H_\Omega^\bullet(M)$. Thus deRham cohomology could be thought of as a *contravariant* functor from the category of $C^\infty$ manifolds to the category of *graded-commutative* $\mathbb{R}$-algebras.

## 2. Partition of unity

In our next lecture we shall use the following fact.

Suppose $M$ is a manifold. Then there exists a collection of $C^\infty$

- functions $f_\alpha : M \to \mathbb{R}$,
- a collection $U_\beta$ of open sets such that $\{U_\beta\}$ form a locally finite covering of $M$ with each $U_\beta$ being a domain of a coordinate chart

such that

1. $f_\alpha \geq 0$;
2. $\operatorname{supp}(f_\alpha) \subset U_\beta$ for some $U_\beta$ in the locally finite covering; and
3. $\sum_\alpha f_\alpha = 1$.