Exterior derivatives and deRham cohomology

1. Examples of exterior derivatives

Let M be a C^∞ manifold of dimension m. Let (U, φ_U) be any coordinate chart on M. Suppose $x_1, \ldots, x_m$ is a collection of local coordinates.

Note that $d(x_i)\left(\frac{\partial}{\partial x_j}\Big|_x\right) = \delta_{i,j}$ and hence $\{d(x_i) \mid 1 \le i \le m\}$ is the dual basis for $\left\{\frac{\partial}{\partial x_j}\Big|_x \mid 1 \le j \le m\right\}$.

Now let $f \in C^\infty(M)$. Then $df = \sum_{i=1}^m a_i dx_i$ for some C^∞ functions a_i on M. Since $a_j(x) = (\sum_{i=1}^m a_i dx_i)\left(\frac{\partial}{\partial x_j}\Big|_x\right) = df(\frac{\partial}{\partial x_j}\Big|_x) = \frac{\partial}{\partial x_j}\Big|_x f$, we have

$$df = \sum_{i=1}^m \left(\frac{\partial}{\partial x_i}\Big|_x f\right) dx_i.$$

Example 1. Let S^1 be the unit circle in $\mathbb{R}^2$. Then for $(x,y) \in S^1$, let $\theta = \arctan(y/x)$. This means that consider $U = \left\{(x,y) \in S^1 \mid x < 1\right\} = S^1 \setminus (1,0)$. Then

$$\theta = \begin{cases} \arctan(y/x) & \text{if } x \neq 0,\ \arctan(y/x) \in (0, 2\pi) \\ \frac{\pi}{2} & \text{if } (x,y) = (0,1) \\ \frac{3\pi}{2} & \text{if } (x,y) = (0,-1) \end{cases}$$

Check that $d\theta = -ydx + xdy$. Similary prove that you get the same expression for the other chart proving that $d\theta$ makes sense. However θ is not a function on S^1. Later on we shall see that $d\theta$ cannot be written as df for any function $f \in C^\infty(S^1)$.

Now we have constructed a sequence

$$\cdots \to \Omega^{p-1}(M) \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \to \cdots$$

Since $d \circ d = 0$, this forms a complex and hence we can define deRham cohomology as follows:

Definition 2. The *p-th deRham cohomology group* of M is the $\mathbb{R}$-vector space

$$H^p_\Omega(M) = \frac{\text{closed } p\text{-forms}}{\text{exact } p\text{-forms}}$$

where

closed p-form: is a form ω such that $d\omega = 0$, and
exact p-form: is a form ω such that there exists η such that $\omega = d\eta$.

Note that the closed p-forms are element of $\ker(d : \Omega^p(M) \to \Omega^{p+1}(M))$ and the exact p-forms are elements of $\operatorname{im}(d : \Omega^{p-1}(M) \to \Omega^p(M))$. Since d are $\mathbb{R}$-linear, there is a natural $\mathbb{R}$ vector space structure on $H^p_\Omega(M)$ for every $\mathbb{R}$.

Also note that all k forms, for $k > m$ are zero. This is because, the basis elements $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ are all zero since two of the subindices are forced to be equal. This means that $H^k_\Omega(M) = 0$ for all $k > m$.

Consider the $\mathbb{R}$ vector space

$$H^\bullet_\Omega(M) = \bigoplus_{i=0}^m H^i_\Omega(M).$$

2. Product structure on $H_{\Omega}^{\bullet}(M)$

One defines a product

$$\wedge : H_{\Omega}^{p}(M) \times H_{\Omega}^{q}(M) \to H_{\Omega}^{p+q}(M)$$

by $[\omega] \wedge [\eta] = [\omega \wedge \eta]$. First note that

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta = 0$$

and hence the above map does map into $H_{\Omega}^{p+q}(M)$. To see that it is well defined, let $\omega, \omega' \in [\omega]$ and $\eta, \eta' \in [\eta]$. Let $\omega - \omega' = d\theta$ and $\eta - \eta' = d\nu$. Also note that $d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^p \omega \wedge d\nu = (-1)^p \omega \wedge d\nu$ and $d(\theta \wedge \eta) = d\theta \wedge \eta$. Thus,

$$\begin{aligned}
\omega \wedge \eta - \omega' \wedge \eta' &= \omega \wedge \eta - \omega \wedge \eta' + \omega \wedge \eta' - \omega' \wedge \eta' \\
&= \omega \wedge (\eta - \eta') + (\omega - \omega') \wedge \eta' = \omega \wedge d\nu + d\theta \wedge \eta \\
&= (-1)^p d(\omega \wedge \nu) + d(\theta \wedge \eta) \\
&= d((-1)^p \omega \wedge \nu + \theta \wedge \eta)
\end{aligned}$$

proving that the map $\wedge$ is well defined on the cohomomology classes.

3. Push-forward of vector fields and pull-back of forms

Let M and N be C^∞ manifolds of dimensions m and n respectively. Let $\Phi : M \to N$ be a C^∞ map. We define the following.

3.1. **Functions.** We define $\Phi^* : C^\infty(N) \to C^\infty(M)$ by $\Phi^*(g) = g \circ \Phi$ for $g \in C^\infty(N)$. Note that for germs of functions at $p \in M$, it induces

$$\Phi^* : C^\infty(\Phi(p)) \to C^\infty(p).$$

3.2. **Tangent vectors.** For $p \in M$ let $X \in T_p(M)$. One defines $\Phi_* X \in T_{\Phi(p)}(N)$ by

$$(\Phi_* X)g = X(\Phi^* g) = X(g \circ \Phi), \text{ for all } g \in C^\infty(\Phi(p)).$$

Note that we are just defining Φ_* for tangent vectors.

Can we do this for vector fields? If not, can we do it after assuming some conditions on Φ?

3.3. **Forms.** Let $\omega \in \Omega^k(N)$. We define $\Phi^* \omega \in \Omega^k(M)$ as follows. For any $p \in M$,

$$(\Phi^* \omega)_p(X_{1,p}, \ldots, X_{k,p}) := \omega_{\Phi(p)}(\Phi_* X_{1,p}, \ldots, \Phi_* X_{k,p})$$

for any $X_{i,p} \in T_p(M)$, $i = 1, \ldots, k$.

Exercise 3. Check that if ω is a C^∞ k-form, then so is $\Phi^* \omega$.