Multilinear algebra, Differential forms and deRham cohomology

1. Multilinear algebra

Let V be a vector space over the real numbers $\mathbb{R}$. A p-form is a multilinear map

$$\omega : \underbrace{V \otimes \cdots \otimes V}_{p \text{ copies}} \to \mathbb{R}$$

such that $\omega(\ldots, v_i, \ldots, v_j, \ldots) = -\omega(\ldots, v_j, \ldots, v_i, \ldots)$. This also means

$$\omega(v_1, \ldots, v_p) = \operatorname{sgn}(\sigma)\omega(v_{\sigma_1}, \ldots, v_{\sigma_p}),$$

where $\sigma \in S_p$ is a permutation and $\sigma_i = \sigma(i)$.

Let us denote the set of all p-forms on a vector space V by $A^p(V)$. One defines $A^0(V)$ to be $\mathbb{R}$. Note that $A^1(V) = V^*$. For $\omega \in A^p(V)$ and $\eta \in A^q(V)$, one defines the wedge product as

$$\omega \wedge \eta(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sgn}(\sigma)\omega(v_{\sigma_1}, \ldots, v_{\sigma_p})\eta(v_{\sigma_{p+1}}, \ldots, v_{\sigma_{p+q}}).$$

Note that the wedge product is **bilinear** and **associative**.

Example 1. $\det : V \otimes \cdots \otimes V \to \mathbb{R}$ where $\det(v_1, \ldots, v_n)$ is the determinant of the matrix whose columns are $v_1, \ldots, v_n$ respectively is an n-form.

Proposition 2. *If $w_1, \ldots, w_n$ is a basis of $V^* = A^1(V)$,*

$$\left\{w_{i_1} \wedge \cdots \wedge w_{i_p} \mid 1 \le i_1 < \cdots < i_p \le n\right\}$$

form a basis for $A^p(V)$.

Corollary 3. *The following facts follow from the above proposition and definitions.*

(1) $\dim A^p(V) = \binom{n}{p}$.
(2) *If $\omega \in A^p(V)$ and $\eta \in A^q(V)$, then $\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega$.*
(3) *If p is odd and $\omega \in A^p(V)$, then $\omega \wedge \omega = 0$.*

2. Differential forms

Let M be an m-dimensional manifold.

Definition 4. A p form on M is an assignment which assigns an element of $A^p(T_y(M))$ for each point $y \in M$.

Now we shall define a differential p-form. For $y \in M$ consider a coordinate chart (U, φ_U) containing y. Suppose the local coordinates are $x_1, \ldots, x_m$. At each point $z \in U$, $\frac{\partial}{\partial x_i}\Big|_{x=z}$, $i = 1, \ldots, m$ form a basis for $T_z(M)$. Let dx_i, $i = 1, \ldots, m$ be the dual basis for $A^1(T_z(M))$. We know that any p-form ω can be written uniquely as

$$\omega_z = \sum_{1 \le i_1 < \ldots < i_p \le n} a_{i_1, \ldots, i_p}(z) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

ω is said to be C^∞ if each of the $a_{i_1, \ldots, i_p}$'s are.

Notation 5. Let $\Omega^p(M)$ be the set of all C^∞ p-forms on M.

3. Exterior differentiation

Consider a 0-form f. Note that 0-forms are nothing but C^∞ functions.

Define a 1-form df as $df(X) = Xf$ for any C^∞ vector field X on M. Let (U, φ_U) be a coordinate chart with local coordinates $x_1, \dots, x_m$. For an r form $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_r}$ on U define

$$d\omega = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_r}.$$

Exercise 6. Check that $d : \Omega^p(U) \to \Omega^{p+1}(U)$ is such that

(1) d is linear,
(2) For $V \subset U$, $d(\omega|_V) = (d\omega)|_V$.
(3) d satisfies the *Leibnitz rule*

$$\omega \in \Omega^r(U),\ \eta \in \Omega^s(U) \implies d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta,$$

(4) For $f \in \Omega^0(U) = C^\infty(U)$, $df(X) = Xf$ for all C^∞ vector fields X, and
(5) For all $f \in \Omega^0(U) = C^\infty(U)$, $d(d(f)) = 0$.

Proposition 7. *For any manifold M, if a $d : \Omega^p(M) \to \Omega^{p+1}(M)$ exists for all p satisfying all the points in the above exercise, then it has to be unique.*

Proof. Suppose d_1 and d_2 are two such external differentials. We prove that $d_1\omega = d_2\omega$ for all $\omega \in \Omega^p(M)$ for all p.

First let $f \in \Omega^0(M) = C^\infty(M)$. Let $d_1 f = \sum_i \alpha_i dx_i$ and $d_2 f = \sum_i \beta_i dx_i$. Evaluating both $d_1 f$ and $d_2 f$ on $\frac{\partial}{\partial x_j}\Big|$, we get $\alpha_j(z) = d_1 f(\frac{\partial}{\partial x_j}\Big|_z) = \frac{\partial}{\partial x_j}\Big|_z f = d_2 f(\frac{\partial}{\partial x_j}\Big|_z) = \beta_j(z)$ proving that $d_1 f = d_2 f$.

Now let ω be an r-form. For any $z \in M$, let (U, φ_U) be a coordinate chart with local coordinates $x_1, \dots, x_m$. Note that, since the coordinate chart is arbitrary, it is enough to prove that $d_1\omega|_U = d_2\omega|_U$. But this is nothing but $d_i(\omega_U)$, $i = 1, 2$, and $\omega_U = \omega|_U$. However, in local coordinates, $\omega_U = \sum_{1 \le i_1 < \cdots < i_r \le n} a_{i_1,\dots,i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}$. Now using Leibnitz rule, we conclude that

$$d_i \omega_U = \sum_{1 \le i_1 < \cdots < i_r \le n} da_{i_1,\dots,i_r} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_r}$$

for $i = 1, 2$, and hence $d_1\omega_U = d_2\omega_U$. □

Exercise 8. Check that the previous proposition and the exercise before that is enough to prove the existence of exterior derivative.