Tangent vectors and vector fields

Today we shall study some properties of tangent spaces, maps between them and introduce vector fields.

1. Some basic properties of the differential

Recall we had defined φ_* for a smooth map $\varphi : M \to N$.

Lemma 1. *For a curve $\gamma : R \to M$, $\gamma_*(\frac{d}{dt}) = \mathscr{D}_\gamma$.*

Proof. As usual, we evaluate at some function $f \in C^\infty(p)$, where $p = \gamma(0)$.

$$\gamma_*(\frac{d}{dt}\Big|_{t=0})f = \frac{d}{dt}\Big|_{t=0} f \circ \gamma$$
$$= \mathscr{D}_\gamma f.$$

□

Lemma 2. *Suppose M and N are C^∞ manifolds of dimensions m and n respectively. Let $F : M \to N$ be a smooth map and let $p \in M$. Consider coordinate charts (U, φ_U) and (V, ψ_V) around p and $F(p)$ respectively. Define*

$$x_j(q) = j\text{-th coordinate of } \varphi_U(q), \text{ and}$$
$$y_i(q') = i\text{-th coordinate of } \psi_V(q'),$$

for $q \in U$ and $q' \in V$. We call $x_j : U \to \mathbb{R}$ and $y_i : V \to \mathbb{R}$ to be the local coordinates *on M and N respectively. Then $F_* : T_p(M) \to T_{F(p)}(N)$ is given by*

$$F_*\left(\frac{\partial}{\partial x_j}\Big|_{x=p}\right) = \sum_{i=1}^{n}\left(\frac{\partial}{\partial x_j}\Big|_{x=p} f_i\right)\frac{\partial}{\partial y_i}\Big|_{y=F(p)}.$$

where $f_i = y_i \circ F$.

Proof. Since $\frac{\partial}{\partial x_j}\Big|_{x=p}$, $j = 1, \ldots, m$ and $\frac{\partial}{\partial y_i}\Big|_{y=F(p)}$, $i = 1, \ldots, n$ are the bases for $T_p(M)$ and $T_{F(p)}(N)$ respectively, we have numbers $a_{i,j}$ such that

$$F_*\left(\frac{\partial}{\partial x_j}\Big|_{x=p}\right) = \sum_{i=1}^{n} a_{i,j}\frac{\partial}{\partial y_i}\Big|_{y=F(p)}.$$

Now to compute $a_{i,j}$, we just evaluate both sides at the function y_k to get

$$F_*\left(\frac{\partial}{\partial x_j}\Big|_{x=p}\right) y_k = \sum_{i=1}^{n} a_{i,j}\frac{\partial}{\partial y_i}\Big|_{y=F(p)} y_k = a_{k,j}$$

since $\frac{\partial}{\partial y_i}\Big|_{y=F(p)} y_k = \delta(k, j)$. This means

$$a_{i,j} = F_*\left(\frac{\partial}{\partial x_j}\Big|_{x=p}\right) y_i = \frac{\partial}{\partial x_j}\Big|_{x=p} y_i \circ F = \frac{\partial}{\partial x_j}\Big|_{x=p} f_i$$

proving the lemma. □

Now we end this section with some useful definitions.

Definition 3. If $F : M \to N$ is a smooth map then F is a/an

immersion: if F_* is one-to-one,

submersion: if F_* is onto at all points.

M is said to be a *submanifold* if F_* is an immersion and F is one-to-one. If $F(M)$ is a submanifold of N and if $F : M \to F(M)$ is a homeomorphism with respect to the subspace topology on $F(M)$ induced from N, then F is said to be an *embedding*.

2. Vector fields

For a smooth manifold M, let TM be the set[1] $\bigcup_{p\in M} T_p(M)$. A vector field on $U \subset M$ is a function $\xi : U \to TU \subset TM$ such that

(1) $\xi(p) \in T_p(M)$, and
(2) ξ is smooth in the sense: for each $p \in U$, consider a chart (W, φ_W) and let the local coordinates be $x_1, \ldots, x_m$. Then for $q \in W$,

$$\xi(q) = \sum_{i=1}^{m} a_i^W(q) \left.\frac{\partial}{\partial x_i}\right|_{x=q}.$$

ξ is *smooth* iff a_i^W are smooth for $i = 1, \ldots, m$ and for every coordinate chart (W, φ_W) on U (belonging to the atlas).

Lemma 4. *ξ is a vector field if and only if for every C^∞ function f on U, the function $p \in U \mapsto \xi(p)f \in \mathbb{R}$ is also a C^∞ function on U.*

The proof is not difficult, but we won't do it now. However you can try it as an exercise.

3. Tangent Bundles

While we are on the subject, before ending the talk, we can as well define tangent bundles.

Consider the set TM we saw above. One can think of TM to be the set

$$TM = \{(p, V) \mid p \in M \text{ and } V \in T_p(M)\}.$$

This description allows us to define a *projection* map $\pi : TM \to M$ by $\pi(p, V) = p$. Note that the *fibre* of π at p is just $T_p(M)$. We shall now give TM a manifold structure.

For each $p \in M$, consider, a coordinate chart (W, φ_W) with local coordinates $x_1, \ldots, x_m$. For each $q \in W$, a tangent vector $V \in T_q(M)$ can be written as $\sum_{i=1}^m a_i \left.\frac{\partial}{\partial x_i}\right|_{x=q}$. Therefore $\pi^{-1}(W)$ can be identified with $W \times \mathbb{R}^n$ where (q, V) is identified with $(q, (a_1, \ldots, a_m))$. Use this identification to give a topology on TM. This, as one can check, gives a manifold structure on TM : $\varphi_W \times id : \pi^{-1}(W) \to \varphi_W(W) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. Thus TM has a manifold structure of dimension $2n$.

[1]It can be given a manifold structure, as we shall see shortly.