Tangent vectors and derivations

Today we shall define tangent vectors of a manifold. I shall first define it as directional derivatives to support our intuition and then show that we can define it more intrinsically as derivations. Finally we shall state some nice theorems based derivations.

1. Tangent vectors and directional derivatives

Let M be a smooth manifold and $p \in M$ be a point. Let $\gamma : \mathbb{R} \to M$ be a smooth curve such that $\gamma(0) = p$.

Definition 1. For a function $f : U \to \mathbb{R}$, for some neighbourhood U of p in M, the *directional derivative of f in the direction of γ at p* is defined by

$$\mathscr{D}_\gamma f = \left.\frac{d}{dt}\right|_{t=0} f \circ \gamma(t).$$

For two curves γ and β with $\gamma(0) = \beta(0) = p$, we say that $\mathscr{D}_\gamma = \mathscr{D}_\beta$ if $\mathscr{D}_\gamma f = \mathscr{D}_\beta f$ for any real valued f defined in a neighbourhood of p. *The set of tangent vectors of M at p* is the set of all directional derivatives upto the above notion of equality.

The fact that f needs to be defined only in some neighbourhood motivates the following definition:

Definition 2. A *germ* of a smooth real valued function f at $p \in M$ on a smooth manifold M is the equivalence class of f under the equivalence relation:

$$f_1 \sim f_2 \iff f_1(x) = f_2(x) \ \forall \ x \in W \text{ for some neighbourhood } W \subset M \text{ of } x.$$

We shall denote the collection of germs at a point p by $C^\infty(p)$.

Note that the collection of germs of functions defined at x, $C^\infty(x)$ naturally form an $\mathbb{R}$-algebra. Furthermore, the tangent vectors (directional derivatives) are defined on germs and they satisfy the following properties:

(1) $\mathscr{D}(af + bg) = a\mathscr{D}f + b\mathscr{D}g$ where a and b are constants; and
(2) $\mathscr{D}(fg) = f(p)\mathscr{D}g + g(p)\mathscr{D}f$.

Definition 3. A function $D : C^\infty(p) \to \mathbb{R}$ is said to be a *derivation at p* if it satisfies (1) and (2) mentioned above.

Proposition 4. *Suppose D is a derivation at p. Then $D = \mathscr{D}_\gamma$ for some γ passing through p.*

Suppose (W, φ_W) be a coordinate chart around p. For $q \in W$, define $x_i(q) =$ i-th coordinate of $\varphi_W(q)$. Also define $\left.\frac{\partial}{\partial x_i}\right|_{x=p} = \mathscr{D}_{\gamma_i}$ where $\gamma_i(t) = \varphi_W^{-1}(p_1, \dots, p_i + t, \dots, p_n)$ where $(p_1, \dots, p_n) = \varphi_W(p)$.

Proof. First we prove that given any $f : U \to \mathbb{R}$ defined in a neighbourhood $U \subset W$ of p, we can find functions $g_1, \dots, g_n : B \to \mathbb{R}$ defined in a smaller neighbourhood B of U such that $\varphi_W(B)$ is a ball around $\varphi_W(p)$, and

(1) $g_i(p) = \left.\frac{\partial}{\partial x_i}\right|_{x=p} f$; and
(2) $f(x) = f(p) + \sum_{i=1}^n (x_i(x) - x_i(p)) g_i(x)$.

By fundamental theorem of calculus, we have

$$\int_0^1 \frac{d}{dt}\bigg|_{t=s} f\circ\varphi_W^{-1}(\varphi_W(p)+t(\varphi_W(x)-\varphi_W(p)))ds = f(x)-f(p).$$

Therefore,

$$f(x)=f(p)+\sum_{i=1}^{n}(x_i(x)-x_i(p))\int_0^1 \frac{\partial}{\partial x_i}\bigg|_{x=\varphi_W^{-1}(\varphi_W(p)+s(\varphi_W(x)-\varphi_W(p)))} f ds$$

Therefore $g_i(s)=\int_0^1 \frac{\partial}{\partial x_i}\Big|_{x=\varphi_W^{-1}(\varphi_W(p)+s(\varphi_W(x)-\varphi_W(p)))} f ds$ satifies the above properties.

Now $Df = 0+\sum_{i=1}^{n} D((x_i(x)-x_i(p)))g_i(p)+0\cdot Dg_i = \sum_{i=1}^{n} D(x_i(x))g_i(p)$. Suppose $\alpha_i = D(x_i(x))$. Then we have

$$Df=\sum_{i=1}^{n}\alpha_i \frac{\partial}{\partial x_i}\bigg|_{x=p} f.$$

Now consider the curve $\gamma(t)=\varphi^{-1}(\varphi_W(p)+(\alpha_1 t,\ldots,\alpha_n t))$.
Check : $\mathscr{D}_\gamma = D$.

This proves the proposition. □

Exercise 5. Check that the space of all derivations form a vector space.

Definition 6. The space of all derivations at p will be called the *tangent space* of M at p and will be denoted by $T_p(M)$.

Corollary 7. *The proof of the above proposition also proves that the space of tangent vectors are generated by* $\frac{\partial}{\partial x_i}\Big|_{x=p}$, *for* $i=1,\ldots,n$.

Exercise 8. Considering the functions x_i prove that the $\frac{\partial}{\partial x_i}\Big|_{x=p}$ are linearly independent.

Therefore, the set $\left\{ \frac{\partial}{\partial x_1}\Big|_{x=p},\ldots,\frac{\partial}{\partial x_n}\Big|_{x=p}\right\}$ form a basis of the space of tangent vectors $T_p(M)$.

2. Functoriality

Suppose $\varphi: M\to N$ be a smooth map. Then one can defines the differential φ_* at a point $p\in M$ as follows:

Definition 9. For $\mathscr{D}_\gamma\in T_p(M)$, $\varphi_*(\mathscr{D}_\gamma)\in T_{\varphi(p)}(N)$ is defined to be $\mathscr{D}_{\varphi\circ\gamma}$.

Proposition 10. φ_* *is linear and it satisfies the property that* $\varphi_* D(f) = D(f\circ\varphi)$. *Also it is functorial in the sense that if* $M\xrightarrow{\varphi} N\xrightarrow{\psi} P$ *are smooth maps between manifolds, then*

$$(\psi\circ\varphi)_* = \psi_*\circ\varphi_*.$$

Proof. By the previous proposition $D=\mathscr{D}_\gamma$ for some γ. Now apply the definition.

$$\begin{aligned}\varphi_* D(f) &= \varphi_*\mathscr{D}_\gamma f = \mathscr{D}_{\varphi\circ\gamma} f\\ &= \frac{d}{dt}\bigg|_{t=0} f\circ\varphi\circ\gamma = \mathscr{D}_\gamma(f\circ\varphi)\\ &= D(f\circ\varphi)\end{aligned}$$

Now the linearity follows as $\varphi_*(aD + bD')f = (aD + bD')(f \circ \varphi) = aD(f \circ \varphi) + bD'(f \circ \varphi) = (a\varphi_* D + b\varphi_* D')f$.

Also $(\psi \circ \varphi)_* Df = D(f \circ \psi \circ \varphi) = \varphi_* D(f \circ \psi) = \psi_* \varphi_* Df$ proving the last equality. $\square$