Orientation, manifolds with boundary, induced structures

1. Orientation on a manifold

Definition 1. An *orientable manifold* M is a manifold with a collection of compatible coordinate charts $\{(U_\alpha, \varphi_{U_\alpha})\}_\alpha$ such that

1. for any pair of coordinate charts $(U_\alpha, \varphi_{U_\alpha})$ and $(U_\beta, \varphi_{U_\beta})$ the Jacobian of the change of coordinate map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ has positive determinant at every point of $\varphi_\alpha(U_\alpha \cap U_\beta)$;
2. U_α cover M.

An *atlas* of M is a maximal collection satisfying the above properties.

Example 2.

Orientable: S^n, torus, connected sum of tori,
Non-orientable: (open) Möbius band, $\mathbb{P}^2$, Klein bottle,

(If you already don't know how to do it, don't worry about proving that these manifolds are non-orientable right now.)

2. Manifold with boundary

We shall need this notion when we do Stokes' theorem.

Let $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \leq 0\}$.

Definition 3. A *differential (or C^∞) manifold of dimension n with boundary* is a topological space M, such that

1. M is Hausdorff,
2. it is second countable,
3. and for every point $p \in M$, there exists a neighbourhood U of p and a morphism $\varphi_U : U \to U' \subset H^n$ such that φ_U is a homeomorphism onto U', an open subset of H^n,
4. and one can choose a C^∞ compatible collection of coordinate charts. Two coordinate charts (U, φ_U) and (V, φ_V) are said to be *compatible, or C^∞-compatible,* if either $U \cap V = \emptyset$, or
$$\varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V)$$
is a diffeomorphism. A maximal C^∞ compatible collection of coordinate charts is, again, called an atlas.

Definition 4. Suppose $p \in M$ be a point on a manifold with boundary. p is said to be a *boundary point* if there exists a chart (U, φ_U) around p such that the first coordinate of $\varphi_U(p)$ is zero.

Exercise 5. Prove that the definition of a boundary point is independent of the choice of a coordinate chart. That is, if (V, φ_V) is another coordinate chart around p, belonging to the same atlas, prove that p is a boundary point with respect to (U, φ_U) if and only if it is a boundary point with respect to (V, φ_V).

Exercise 6. Given an atlas on a manifold with boundary, M, construct an atlas on the boundary points

$$\partial M = \{p \in M \mid p \text{ is a boundary point of } M\}$$

proving that it is an $n-1$ dimensional manifold. (*Hint :* Use the fact that p has a coordinate chart (U, φ_U) such that $\varphi_U(p) = (0, \dots)$. Check that one can define $\varphi'_U : U \cap \partial M \to \partial U'$ and these form an atlas.)

3. Smooth morphims

Definition 7. A map $f : M \to N$ between two manifolds is said to be C^∞ or *smooth* if for any $p \in M$ and a chart (U, φ_U) on M containing p and for any chart (V, ψ_V) containing $f(p)$ on N, $\varphi_V \circ f \circ \varphi_U^{-1} : \varphi_U(U \cap f^{-1}(V)) \to \psi_V(V)$ is a smooth map.

Exercise 8. Prove that $f : M \to N$ is smooth if and only if $f : (M, \mathscr{F}_M) \to (N, \mathscr{F}_N)$, where $\mathscr{F}_X = C^\infty(X)$.

4. Induced structures

Definition 9. Suppose $(X, \mathscr{F}_X)$ is a functional structured space. Let $\varphi : X \to Y$ be a map. Then the induced functional structure on Y is given by

$$\mathscr{F}_Y(U) = \left\{ f : U \to \mathbb{R} \mid f \circ \varphi \in \mathscr{F}_X(\varphi^{-1}(U)) \right\}.$$

Example 10. Consider $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be the projection (to the first n coordinates). Suppose $\mathscr{F}_{\mathbb{R}^m} = C^\infty$. What is the induced structure on $\mathbb{R}^n$?

Note that

$$\mathscr{F}_{\mathbb{R}^n}(U) = \left\{ f : U \to \mathbb{R} \mid g(x, y) = f(x), x \in U, y \in \mathbb{R}^{m-n} \text{ is } C^\infty \right\}.$$

Now check that the l-th partial derivatives of g and f exist and are continuous iff they exist and are continuous for the other one.

Example 11. (1) One can define a *torus* $\mathbb{T}^2$ as $\mathbb{R}^2/\sim$ where $\sim$ is the equivalence relation on $\mathbb{R}^2$ where $(x, y) \sim (u, v)$ if and only if $x - u \in \mathbb{Z}$ and $y - v \in \mathbb{Z}$. We prove that $\mathbb{T}^2$ is a manifold, first by giving a chart, and second by giving a functional structure.

Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the quotient map. For a point $p \in \mathbb{T}^2$, let U be $\pi(B_{1/2}(q))$ for any $q \in \mathbb{R}^2$ such that $\pi(q) = p$. Note that π is a homeomorphism from $B_{1/2}(q)$ to U and hence $(U, \pi^{-1} : U \to B_{1/2}(q))$ is a coordinate chart.

Exercise 12. Check that this coordinate chart is C^∞ compatible and hence defines a manifold structure on $\mathbb{T}^2$. Does this chart also give an orientation?

To give a functional structure, consider $(\mathbb{T}^2, \mathscr{F}_{\mathbb{T}^2})$, the structure induced from $(\mathbb{R}^2, C^\infty_{\mathbb{R}^2})$ on $\mathbb{T}^2$ by π. We want to show that this structure is locally isomorphic to $(U', C^\infty_{U'})$ for some open subset $U' \subset \mathbb{R}^2$. Let $U \subset \mathbb{T}^2$ be as above $(\pi(B_{1/2}(q)))$. Then

$$\mathscr{F}_{\mathbb{T}^2}(U) = \left\{ f : U \to \mathbb{R} \mid f \circ \pi \in C^\infty_X(\pi^{-1}(U)) \right\}.$$

What is $\pi^{-1}(U)$? It is a disjoint union of translations of $B_{1/2}(q)$. Check that $f \circ \pi$ is a $\mathbb{Z} \oplus \mathbb{Z}$ periodic function and it is determined by its value in $B_{1/2}(q)$. Therefore, $\mathscr{F}_{\mathbb{T}^2}(U) = C^\infty_{\mathbb{R}^2}(B_{1/2}(q))$.

Check : $(U, \mathscr{F}_U) \cong (B_{1/2}(q), C^\infty_{B_{1/2}(q)})$.

Hence this gives a manifold structure according to the second definition.

(2) *Reading exercise :* Manifold structure on a sphere S^2 and a projective plane $\mathbb{P}^2$.

5. Pull backs of functional structures

Suppose $(X, \mathscr{F}_X)$ is a functional structure on X. And suppose $\alpha : Y \to X$ is a continuous map. One can define a functional structure on Y as follows for $U \subset X$,

$$\mathscr{F}_Y(\alpha^{-1}(U)) = \left\{ f : \alpha^{-1}(U) \to \mathbb{R} \mid \begin{array}{r} \forall\ p \in \alpha^{-1}(U), \exists \text{ neighbourhood} \\ W \subset X, \alpha(p) \in W \text{ and a } g \in F_X(W), \\ \text{such that } f|_{\alpha^{-1}(W)} = g|_W \circ \alpha \end{array} \right\}.$$

Note that this gives us ways to define functional structures on topological spaces which are not "smooth", like cones etc.