Differentiable manifolds

1. Aim of the course

This course is all about *cohomology.* We shall introduce a couple of forms of cohomology theories, explore methods to compute them, and study their properties. In the second half of the course we shall be concerned with the relationship between cohomology and homology and study some mathematical structures on cohomology.

We shall begin the course by studying de Rham cohomology. Hence we start with differetiable manifolds and then go on to define tangent and cotangent spaces and differential forms. This lecture will introduce the notion of differential manifolds.

2. Differential manifolds: 1st approach

Definition 1. A *differential (or C^∞) manifold of dimension n* is a topological space M, such that

(1) M is Hausdorff,
(2) it is second countable,
(3) and it is *locally Euclidean.*

By "locally Euclidean" we mean the following: for every point $p \in M$, there exists a neighbourhood U of p and a morphism $\varphi_U : U \to U' \subset \mathbb{R}^n$ such that φ_U is a homeomorphism onto U', an open subset of $\mathbb{R}^n$.

Such a pair (U, φ_U) is called a *coordinate chart.* A topological space along with the data about coordinate charts is called a *topological manifold.*

For a topological manifold to be differentiable, we need it to have an atlas, which we shall explain now. Two coordinate charts (U, φ_U) and (V, φ_V) are said to be *compatible, or C^∞-compatible,* if either $U \cap V = \emptyset$, or

$$\varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V)$$

is a diffeomorphism. An *atlas* is a maximal collection of compatible coordinate charts. Here maximal, by definition, means that if $\{(U_\alpha, \varphi_{U_\alpha})\}$ is an atlas and if (V, ψ) is a coordinate chart which is compatible with every $(U_\alpha, \varphi_{U_\alpha})$, then $(V, \psi) \in \{(U_\alpha, \varphi_{U_\alpha})\}$.

A topological manifold M with an atlas is called a *differentiable manifold.*

Definition 2. Suppose $U \subset M$ be an open subset of M. A function $f : U \to \mathbb{R}$ is said to be differentiable (C^∞), if for all coordinate charts (V, φ_V) in the atlas, $f \circ \varphi_V^{-1} : \varphi(U \cap V) \to \mathbb{R}$ is C^∞.

Example 3. Dimension 0 : discrete set of finite or countably many points.
Dimension 1 : $\mathbb{R}$, circle.
Dimension 2 : $\mathbb{R}^2$, sphere, torus, genus g surfaces, $\mathbb{P}^2$, Klein bottle.

3. Differential manifolds: 2nd approach

Many times we are more interested in the space of all functions from a manifold, than the manifold itself. It turns out looking at all the functions is equivalent to looking at the manifold. The rest of the talk will be a clarification of this idea.

X be a topological space. For an open set $U \subset X$, let $C^0(U)$ be the space of all continuous maps from U to $\mathbb{R}$. Note that this set has a natural $\mathbb{R}$-algebra structure.

We construct a category $\mathcal{O}\mathfrak{pen}\mathcal{S}\mathfrak{et}$ whose objects are open subsets of X and for two open subsets U and V of X,

$$\hom(U, V) = \begin{cases} \emptyset & \text{if } U \not\subseteq V, \\ \{\iota\} & \text{if } \iota : U \hookrightarrow V \text{ is the inclusion.} \end{cases}$$

Let us denote the category of all $\mathbb{R}$-algebras by $\mathbb{R}$-$\mathcal{A}\mathfrak{lgebra}$.

Definition 4. A *functional structure on* X is a **functor**

$$\mathscr{F}_X : \mathcal{O}\mathfrak{pen}\mathcal{S}\mathfrak{et} \to \mathbb{R}\text{-}\mathcal{A}\mathfrak{lgebra}$$

such that

(1) $\mathscr{F}_X(U)$ is a subalgebra of $C^0(U)$,
(2) $\mathscr{F}_X(U)$ contains all the constant functions on U,
(3) For all $V \subset U$ and for all $f \in \mathscr{F}_X(U)$, $f|_V \in \mathscr{F}_X(V)$.
(4) If $U = \bigcup_\alpha U_\alpha$, $f \in C^0(U)$ such that $f|_{U_\alpha} \in \mathscr{F}_X(U_\alpha)$ for all α, then $f \in \mathscr{F}_X(U)$.

The pair $(X, \mathscr{F}_X)$ is called a *functionally structured space.*

Exercise 5. Check that the following are functional structures on $\mathbb{R}^n$:

(1) $\mathscr{F}_{\mathbb{R}^n}(U) = C^0(U)$,
(2) $\mathscr{F}_{\mathbb{R}^n}(U) = C^k(U)$,
(3) $\mathscr{F}_{\mathbb{R}^n}(U) = C^\infty(U)$,
(4) $\mathscr{F}_{\mathbb{R}^n}(U) = C^\omega(U)$.

Now we come to the definition of a manifold.

Definition 6. A morphism of two functionally structured spaces $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ is a continuous map $\varphi : M \to N$ such that for any $U \subset N$, $f \in \mathscr{F}_N(U)$, $f \circ \varphi \in \mathscr{F}_M(\varphi^{-1}(U))$.

Exercise 7. Suppose $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, and $\varphi : U \to V$ is a continuous map.

(1) Prove that if φ is C^∞, it induces a morphism
$$\varphi : (U, C^\infty(U)) \to (V, C^\infty(V)).$$
(2) Suppose the induced map
$$\varphi : (U, C^0(U)) \to (V, C^\infty(V)).$$
actually maps to $(V, C^\infty(V))$. Show that φ is C^∞.

Notation 8. Suppose $(X, \mathscr{F}_X)$ is a functionally structured space. Suppose $U \subset X$ be an open subset. Then for an open subset V of U, define

$$\mathscr{F}_U(V) = \mathscr{F}_X(V).$$

This makes $(U, \mathscr{F}_U)$ a functionally structured space.

Definition 9. An n-dimensional differentiable manifold M is a

(1) Hausdorff,
(2) second countable,
(3) functionally structured space $(M, \mathscr{F}_M)$ such that for each point $p \in M$, there exists a neighbourhood U of p, an open set U' of $\mathbb{R}^n$ and an isomorphism $\varphi : (U, \mathscr{F}_U) \to (U', \mathscr{F}_{U'})$.

Exercise 10. Prove that the two definitions of differentiable manifolds are equivalent.