17/03/2013

de Rham's theorem :

Consider a p-form ω and a smooth simplex $\sigma : \Delta_p \to M$

Define $\displaystyle\int_\sigma \omega = \int_{\Delta_p} \sigma^* \omega$

Let $C_p^\infty(M)$ be the collection of C^∞ maps from $\Delta_p \to M$.

We had a boundary map $\partial_p : C_p^\infty(M) \to C_{p-1}^\infty(M)$

such that $\partial_p \circ \partial_{p+1} = 0 \quad \forall p$.

Define $\quad C_\infty^p(M) = \mathrm{Hom}(C_p^\infty(M), \mathbb{R})$

Let $\delta_p : C_\infty^p(M) \to C_\infty^{p+1}(M)$. be the dual map $(\_ \circ \partial_{p+1})$

we also defined a map $\Psi_p : \Omega^p(M) \to C_\infty^p(M)$

$$\omega \longmapsto \left[\sigma \mapsto \int_\sigma \omega\right]$$

we had proved that

$$\boxed{\Psi_p(d\omega) = \delta \Psi_{p-1}(\omega)}$$

$\therefore$ $\Psi_\bullet$ is a chain homomorphism : $\Omega^\bullet(M) \to C_\infty^\bullet(M)$

Therefore we get a map

$$\Psi^* \; \cancel{\Psi^*} : H_\Omega^p(M) \longrightarrow H_{\text{smooth}}^p(M, \mathbb{R})$$

deRham cohomology

<u>THM</u>: (de Rham): The homomorphism

$$\Psi^*: H^p_{dR}(M) \longrightarrow H^p_{sm}(M, \mathbb{R})$$

is an isomorphism $\forall p$ and $\forall$ smooth manifolds M.

REDUCTION LEMMA:

We use a lemma to reduce to proving deRham's thm to convex open sets:

<u>Lemma</u> (9.5) Let M^n be a smooth n-manifold. Suppose $P(U)$ is a statement about open sets in M satisfying

1. $P(U)$ is TRUE for convex open $U \subset \mathbb{R}^n$
2. $P(U)$ & $P(V)$ & $P(U \cap V)$ all TRUE $\Rightarrow$ $P(U \cup V)$ is true
3. $\{U_\alpha\}$ be disjoint open sets & $P(U_\alpha)$ is TRUE $\forall \alpha \Rightarrow$ $P(\cup U_\alpha)$ is true.

THEN $P(M)$ is true.

<u>Proof</u>: We use a similar argument twice:

first time to prove $P(U)$ is TRUE for U open subsets of $\mathbb{R}^n$

second time (by replacing (1) above by $P(U)$ true for all open) to conclude that $P(M)$ is true

For sake of rigor let us replace (1) above by ~~replacing~~

(1) $P(U)$ is true for all open sets in a class $\mathcal{C}$; $\mathcal{C}$ form a basis ~~is closed under finite intersections~~. for the topology

and replace the conclusion by

$P(W)$ is true where W can be written as a COUNTABLE union of precompact open sets from $\mathcal{C}$.

STEP I $P(V)$ is true where $V = U_1 \cup \dots \cup U_n$ with $U_i \in \mathcal{C}$, $i=1,\dots,n$.

Pf: (INDUCTION)

$n=2$: ① $\Rightarrow$ True for $P(U_1)$, $P(U_2)$ and $P(U_1 \cap U_2)$

② $\Rightarrow$ True for $P(U_1 \cup U_2)$

Suppose true for $n=k$

Ind hyp $\Rightarrow P(U_1 \cup \dots \cup U_k)$ is True

$(U_1 \cap U_{k+1}), \dots, (U_k \cap U_{k+1}) \in \mathcal{C} \underset{\text{Ind hyp}}{\Rightarrow} P((U_1 \cap U_{k+1}) \cup \dots \cup (U_k \cap U_{k+1}))$ is True

$\Rightarrow P((U_1 \cup \dots \cup U_k) \cap U_{k+1})$ is True.

$P(U_k)$ is anyway TRUE

$\therefore$ ② $\Rightarrow P((U_1 \cup \dots \cup U_k) \cup U_{k+1})$ is TRUE

This completes STEP I

Now let $W = \bigcup_{i=1}^{\infty} U_i$, a countable union of precompact open sets in $\mathcal{C}$

STEP II Construction of a proper map $f : W \to [0, \infty)$.

[Defn: ~~Inverse image of a compact set is compact~~.
$f : X \to Y$ is proper if the inverse image of a compact set is compact]

Note that $W = \bigcup_{i=1}^{n} U_i$ s.t. $\overline{U_i}$ is compact.

Suppose $\{(U_i, f_i)\}$ be a partition of unity ~~on this subs~~ on W.

Define $f(x) = \sum_n n f_n(x)$.

Ex: Use the fact that f_n is a partition of unity to prove that f is a proper map.

STEP III : $P(W)$ is TRUE

Define $A_n = f^{-1}([n, n+1]) \subset W$.

A_n is compact, since f is proper.

Since $A_n \subset f^{-1}((n-\frac{1}{2}, n+\frac{3}{2}))$, $\exists$ finitely many open sets $O_1^n, \dots, O_{m_n}^n$ ~~O_{m_n}~~ of $\mathcal{C}$ s.t. $A_n \subset \bigcup_{i=1}^{m} O_i^n$. Let $X_n = \bigcup O_i^n$.

From step I, $P(X_n)$ is true and

$$A_n \subset X_n \subset f^{-1}((n-\tfrac{1}{2}, n+\tfrac{3}{2})).$$

$\Rightarrow X_{2n} \cap X_{2m} = \emptyset$ and $X_{2n-1} \cap X_{2m-1} = \emptyset \quad \forall m \neq n.$

Let $A = \bigcup_n X_{2n}$ $\quad B = \bigcup_n X_{2n-1}$.

By assumption ③, $P(A)$ & $P(B)$ are true.

Moreover, $A \cap B = \cancel{\coprod (X_{2t} \cap X_{2s-1})}\ \coprod (X_{2t} \cap X_{2s-1})$

and each ~~$X_{2t} \cap X_{2s-1}$~~ $X_{2t} \cap X_{2s-1} = (O_1^{2t} \cup \dots \cup O_{m_{2t}}^{2t}) \cap (O_1^{2s-1} \cup \dots \cup O_{m_{2s-1}}^{2s-1})$

is a finite union of open sets in $\mathcal{C}$

$\Rightarrow P(X_{2t} \cap X_{2s-1})$ is TRUE

③ $\Rightarrow P(A \cap B)$ is TRUE

$\Rightarrow P(A \cup B)$ is TRUE

But ~~$P(A \cup B)$~~ $A \cup B \supset f^{-1}([0, \infty)) = W$

~~know~~ & $A \cup B \underset{\text{by def}}{\subset} f^{-1}([0,\infty)) = W$

$\Rightarrow W = A \cup B$ $\square$.

To conclude the proof, note that convex open sets form a basis of M, and any open set can be written as an union of countably many precompact convex open sets

$\Rightarrow P(U)$ is true for ~~conv~~ all open sets

Now any ~~manifold~~ second countable manifold can be written as a countable union of precompact open sets

$\Rightarrow P(M)$ is true $\square$

For us P will be de Rham's Thm:

In view of the above result, we reduce to proving

1. ~~Dex~~ de Rham's thm ~~is~~ holds for convex open sets
2. will be implied by Mayer Vietoris on either side
3. " " " " ~~same property~~ additivity.

---

1 Lemma (Poincaré lemma) The de Rham's thm is true for any convex open subset $U \subset \mathbb{R}^n$.

Pf: Check that for a convex open set, the singular cohomology is given by

$$\cancel{H^p(U,\mathbb{R})} \; H^p(U,\mathbb{R}) = \begin{cases} \mathbb{R} & \text{for } p=0 \\ 0 & \text{o.w.} \end{cases}$$

Thus we have to prove that

(a) $H^0_\Omega(U) = \mathbb{R}$; i.e. for any smooth function f on U, $df=0 \Rightarrow f$ is constant

& $\Psi^*: \mathbb{R} \to \mathbb{R}$ is ~~the~~ an isomorphism.

(b) $H^p_{\Omega}(M)=0 \quad \forall\, i \geq 1$; i.e. every closed p form on U is exact.

<u>Proof:</u> (a) $df=0 \Rightarrow \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i = 0 \Rightarrow \frac{\partial f}{\partial x_i} = 0 \ \forall\, i$

$\Rightarrow f$ is a constant. ($\because U$ is connected)

The second part of (a) follows from the fact that

$$\psi^*(r)(\sigma) = r \qquad \forall\ \sigma : pt. \longrightarrow U.$$

(b) This involves a trick:

Define $\varphi : \Omega^{p+1} \longrightarrow \Omega^{p}$ by

$$\omega = f(x_1, \dots, x_n)\, dx_{j_1} \wedge \dots \wedge dx_{j_p} \longmapsto \cancel{\phi(\omega) = \int_0^1 t^p f(tx)}$$

$$\varphi(\omega) = \left(\int_0^1 t^p f(tx_1, \dots, tx_n)\, dt \right) \eta$$

where $\eta = \sum_{i=0}^{p} (-1)^i x_{j_i}\, dx_{j_0} \wedge \dots \wedge \widehat{dx_{j_i}} \wedge \dots \wedge dx_{j_p}$.

<u>Want:</u> $\omega = d\varphi(\omega) + \varphi(d\omega)$.

$$d\varphi(\omega) = d\left(\int_0^1 t^p f(tx_1, \dots, tx_n)\, dt \right) \wedge \eta + \left(\int_0^1 t^p f(tx_1, \dots, tx_n)\, dt \right) d\eta$$

$$\cancel{= \left(\int_0^1 \left[t^p \cdot t \sum_{k=1}^{n} \left(t \frac{\partial f}{\partial x_k} (tx_1, \dots, tx_n) \right] dt \right) \eta}$$

$$= \left(\int_0^1 t^p \left[\sum_{k=1}^{n} \frac{\partial f}{\partial x_k} (tx_1, \dots, tx_n) \cdot t \right] dx_k\, dt \right) \wedge \eta + \left(\int_0^1 t^p f(tx_1, \dots, tx_n)\, dt \right) d\eta$$

$$= \underbrace{\left(\sum_{k=1}^{n} \int_0^1 t^{p+1} \frac{\partial f}{\partial x_k} (tx_1, \dots, tx_n)\, dt \right) dx_k \wedge \eta}_{S} + \underbrace{\left(\int_0^1 t^p f(tx_1, \dots, tx_n)\, dt \right) \eta}_{T}$$

Moreover,

$$d\omega = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\, dx_k \wedge dx_{j_0} \wedge \dots \wedge dx_{j_p}$$

$$\Rightarrow \varphi(d\omega) = \sum_{k=1}^{n} \varphi\left(\frac{\partial f}{\partial x_k}\, dx_k \wedge dx_{j_0} \wedge \dots \wedge dx_{j_p}\right)$$

$$= \sum_{k=1}^{n} \left(\int_0^1 t^{p+1} \frac{\partial f}{\partial x_k}(tx_1,\dots,tx_n)\, dt\right)\left(x_k\, dx_{j_0} \wedge \dots \wedge dx_{j_p} - \sum_{i=0}^{p} (-1)^i x_{j_i}\, dx_k \wedge dx_{j_0} \wedge \dots \wedge \widehat{dx_{j_i}} \wedge \dots \wedge dx_{j_p}\right)$$

$$= \sum_{k=1}^{n} \left(\int_0^1 t^{p+1} \frac{\partial f}{\partial x_k}(tx_1,\dots,tx_n)\, dt\right)\left(x_k\, dx_{j_0} \wedge \dots \wedge dx_{j_p} - dx_k \wedge \eta\right)$$

$$= \sum_{k=1}^{n} x_k \left(\int_0^1 t^{p+1} \frac{\partial f}{\partial x_k}(tx_1,\dots,tx_n)\, dt\right) dx_{j_0} \wedge \dots \wedge dx_{j_p}$$

$$- \sum_{k=1}^{n} \left(\int_0^1 t^{p+1} \frac{\partial f}{\partial x_k}(tx_1,\dots,tx_n)\, dt\right) dx_k \wedge \eta$$

$$= \left(\int_0^1 t^{p+1} \left(\sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(tx_1,\dots,tx_n)\, x_k\right) dt\right) dx_{j_0} \wedge \dots \wedge dx_{j_p} - S$$

$$= \left(\int_0^1 t^{p+1} \left(\frac{d}{dt} f(tx_1,\dots,tx_n)\right) dt\right) dx_{j_0} \wedge \dots \wedge dx_{j_p} - S$$

$$= \left[\left[t^{p+1} \cdot f(tx_1,\dots,tx_n)\right]_0^1 - \underline{\int_0^1 (p+1)t^p f(tx_1,\dots,tx_n)\, dt} \right] dx_{j_0} \wedge \dots \wedge dx_{j_p} - S$$

$$= \left[f(x_1,\dots,x_n) - (p+1)\int_0^1 t^p f(tx_1,\dots,tx_n)\, dt \right] dx_{j_0} \wedge \dots \wedge dx_{j_p} - S$$

$$= \omega - T - S \quad (\because d\eta = (p+1)\, dx_{j_0} \wedge \dots \wedge dx_{j_p})$$

Therefore $\omega = d\varphi(\omega) + \varphi(d\omega)$

$\Rightarrow$ whenever $d\omega = 0$, $\omega = d\varphi(\omega)$ and hence ω is exact!

This finishes the proof of Poincaré Lemma $\square$

MAYER-VIETORIS

First we shall check that the following is a short exact sequence:

$$0 \to \Omega^p(U \cup V) \to \Omega^p(U) \oplus \Omega^p(V) \to \Omega^p(U \cap V) \to 0$$

First let us describe the maps

For this consider the open cover $\{U, V\}$ of $U \cup V$

$\Rightarrow \exists$ a locally finite refinement Y_α of $\{U, V\}$ and a collection of functions $\{f_\alpha\}$ with $\mathrm{supp}(f_\alpha) \subset Y_\alpha$

s.t. $\sum f_\alpha(x) = 1 \ \forall x \in U \cup V$

Now ~~each~~ let $f(x) = \sum_{Y_\alpha \subset U} f_\alpha(x)$

$\Rightarrow$ For $x \in U \setminus V$, $f(x) = 1$, and

for $x \in V \setminus U$, $f(x) = 0$.

~~Define $\to f\omega$~~

$\Rightarrow$ For $\omega \in \Omega^p(U \cup V)$, $f\omega$ will be supported on U

and $(1-f)\omega$ will be supported on V

Define

$$\Omega^p(U \cup V) \xrightarrow{i} \Omega^p(U) \oplus \Omega^p(V)$$

by

$$\omega \longmapsto (\omega|_U , -\omega|_V)$$

and $\Omega^p(U) \oplus \Omega^p(V) \xrightarrow{p} \Omega^p(U \cap V)$

by

$$(\omega_U , \omega_V) \longmapsto (\omega_U + \omega_V)|_{U \cap V}$$

Now $\omega|_U = 0$ & $\omega|_V = 0 \Rightarrow \omega = 0$

$\Rightarrow$ The first map is injective.

Further $(\omega_U + \omega_V)|_{U \cap V} = 0 \Rightarrow \omega_U|_{U\cap V} = -\omega_V|_{U \cap V}$

$\Rightarrow \omega_U$ and $-\omega_V$ patch to give a p-form ω on $U \cup V$

$\Rightarrow \operatorname{Ker} p \subset \operatorname{Im} i$

$\operatorname{Im} i \subset \operatorname{Ker} p$ since $p \circ i(\omega) = \omega|_{U\cap V} - \omega|_{U \cap V} = 0$.

$\Rightarrow \operatorname{Ker} p = \operatorname{Im} i$

We only need p is surj

Choose ω on $U \cap V$. Extend $f\omega$ to U and $(1-f)\omega$ to V

Then $p(f\omega, (1-f)\omega) = \omega$.

R

This immediately gives us the Mayer Vietoris long exact sequence for deRham cohomology

$$0 \to H^p_\Omega(U\cup V) \to H^p_\Omega(U) \oplus H^p_\Omega(V) \to H^p_\Omega(U\cap V) \xrightarrow{\delta} H^{p+1}_\Omega(U\cup V) \to \cdots$$

~~Suppose deRham's~~

we also have a ladder induced by Ψ

$$\begin{tikzcd}
0 \arrow[r] & H^p_\Omega(U\cup V) \arrow[r] \arrow[d, "\Psi"] & H^p_\Omega(U) \oplus H^p_\Omega(V) \arrow[r] \arrow[d, "\Psi\oplus\Psi", "\simeq"'] & H^p_\Omega(U\cap V) \arrow[r] \arrow[d, "\Psi \cong"] & H^{p+1}_\Omega(U\cup V) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & H^p(U\cup V;\mathbb{R}) \arrow[r] & H^p(U;\mathbb{R}) \oplus H^p(V;\mathbb{R}) \arrow[r] & H^p(U\cap V;\mathbb{R}) \arrow[r] & H^{p+1}(U\cup V;\mathbb{R}) \arrow[r] & \cdots
\end{tikzcd}$$

$\Rightarrow$ If deRham's thm is true for U, V & $U\cap V$, it is true for $U\cup V$.

This was 2 $\square$

Disjt Union: Exercise.

REVIEW OF CW complexes

18/03/2013

- Review of ~~cellular~~ CW complexes
- Review of cellular homology
- Cellular approximation theorem
- Formula for the map induced on the cellular chain complex from a cellular map
- Statement of Hopf Classification Theorem
- UCT $\Rightarrow$ Singular ~~hom~~ cohomology = Cellular cohomology

Understanding cohomology - (I)

THEOREM: (Hopf Classification theorem)

Let K be a CW complex and assume that $\dim K = n$ or that $n=1$. Then there is a one-to-one correspondence

$$[K; S^n] \leftrightarrow H^n(K; \mathbb{Z})$$

given by $[\varphi] \longleftrightarrow \xi_\varphi$ (to be explained).

Before we even say what ξ_φ is, let us contemplate a bit about $H^n(K;\mathbb{Z})$. We have defined it as singular cohomology. However one can look at the complex defining its CW-homologies:

RECALL: ~~The ho~~

$K^{(0)}$: Set of discrete pts. (0 cells)

$K^{(1)}$: $\{ f_{\partial\sigma} : S^0 \longrightarrow K^{(0)} \}$ be a bunch of maps. $Y^{(1)} = \coprod_\sigma D^1_\sigma$, $B = \coprod_\sigma \partial D^1_\sigma$, $f : B \longrightarrow K^{(0)}$ defined by $f|_{\partial D_\sigma} = f_{\partial\sigma}$

$$K^{(1)} = K^{(0)} \cup_f Y^{(1)}.$$

$f_{\partial\sigma}$ is called the attaching map for $\partial\sigma$

$K^{(n)}$: $\{ f_{\partial\sigma} : S^{n-1} \longrightarrow K^{(n-1)} \}$, $Y^{(n)} = \coprod_\sigma D^n_\sigma$, $B = \coprod_\sigma \partial D^n_\sigma$

$f : B \longrightarrow K^{(n-1)}$ s.t. $f|_{\partial D^n_\sigma} = f_{\partial\sigma}$

$$K^{(n)} = \cancel{K^{(n)}}\, K^{(n-1)} \cup_f Y^{(n)} \quad \ldots \quad \square$$

$K = \cup K^{(n)}$.

Suppose $f_\sigma : D^n_\sigma \to K$ be the canonical map

$$p_\sigma : K^{(n)} \longrightarrow K^{(n)}/K^{(n-1)} \cong \bigvee S^n_\tau \to S^n_\sigma$$

be the projection

For an n-cell σ and an $(n-1)$-cell τ, one defines

$$[\tau : \sigma] = \deg(p_\tau \circ f_{\partial\sigma})$$

$C_n(K) =$ free abelian group ~~of~~ on n-cells

$$\partial\sigma = \sum_{\tau : (n-1)\text{cells}} [\tau:\sigma]\, \tau$$

This complex computes cellular homology (with $\mathbb{Z}$) coeff and it matches with singular homology.

Define $C^n(K;G) = \mathrm{Hom}(C_n(K), G)$

& $(\delta c)(\sigma_{n+1}) = c(\partial\sigma_{n+1})$.

$$= \sum_\tau [\tau : \sigma]\, c(\tau).$$

<u>Example:</u> S^n is a cell complex with cells $*$ (dim 0), e_n (dim n)

D^{n+1} " " " " a single cell e_{n+1}.

$$\gamma_n : \underbrace{I \times \cdots \times I}_{n \text{ tim}} \to S^p \cong (I^p/\partial I^p).$$

~~20~~ 19/03/2013

AIM TO PROVE: (Hopf classification thm)

K - CW complex ; suppose $\dim K = n$

Then there is a 1-1 correspondence

$$[K; S^n] \leftrightarrow H^n(K; \mathbb{Z})$$

Construction of a map between these sets:

First we observe the following 2 statements from yesterday's discussion:

Ⓐ Any cont map $\varphi : K \longrightarrow S^n$ is homotopic to one that takes $K^{(n-1)}$ to $*$.

Ⓑ If $\varphi, \psi : K \longrightarrow S^n$ are two continuous maps, homotopic to each other, then they are homotopic via a homotopy $K \times I \longrightarrow S^n$ which takes

$$(K \times I)^{(n-1)} = (K^{(n-1)} \times \partial I) \cup (K^{(n-2)} \times I) \longrightarrow *$$

Also recall from yesterday:

$f_{\partial\sigma}$ were the attaching maps $\partial\sigma \cong S^{n-1} \longrightarrow K^{(n-1)}$

f_σ is the induced map $D^n_\sigma \longrightarrow K^{(n)} \hookrightarrow K$.

- $(X, x), (Y, y)$ top. sp with base pts
- ~~X~~ $X \amalg Y$ = disjt union of X & Y
- $X \vee Y = \dfrac{X \amalg Y}{x \sim y}$
- $X \wedge Y = \dfrac{X \times Y}{X \vee Y}$

If X, Y are compact, $X \wedge Y$ turns out to be the one-point compactification of $(X \setminus \{x\}) \times (Y \setminus \{y\})$.

For example, $S^p \wedge S^q \cong S^{p+q}$.

- $I = [0, 1]$, $* = 0$, $I^n = I \times \dots \times I$ (n times)

Let $\gamma_1 : I^1 \longrightarrow S^1 \cong I^1/\partial I^1$

$$\gamma_p = \underbrace{I^1 \times \dots \times I^1}_{p \text{ times}} \longrightarrow \underbrace{S^1 \wedge \dots \wedge S^1}_{p \text{ times}} \cong S^p.$$

~~$p_\sigma : I^n \cong D^n \xrightarrow{f_\sigma}$~~

$$p_\sigma : K^{(n)} \longrightarrow \frac{K^{(n)}}{K^{(n-1)}} \cong \bigvee_{\sigma: n\text{cell}} S^n_\sigma \xrightarrow{\text{projection}} S^n_\sigma$$

Note $p_\sigma \circ f_\sigma : I^n \cong D^n \xrightarrow{f_\sigma} K^{(n)} \xrightarrow{p_\sigma} S_\sigma$

is ~~noting~~ nothing but γ_n

Note $\quad p_\sigma \circ f_\sigma = \gamma_n$

$\qquad p_\sigma \circ f_\tau$ = constant map to the base pt. for $\sigma \neq \tau$.

$$[\tau:\sigma] = \deg(p_\tau \circ f_{\partial\sigma})$$

- For a cellular map $g : K \longrightarrow L$

  τ n-cell in L, σ n-cell in K,

  ~~$g_{\tau\sigma} : S^n \to$~~

$$\begin{tikzcd}
\mathbb{D}^n \arrow[r, "f_\sigma"] \arrow[d, "r_n"] & K^{(n)} \arrow[r, "g"] \arrow[d] & L^{(n)} \arrow[r, "p_\tau"] \arrow[d] & S^n \arrow[d, equal] \\
S^n \arrow[r, "\bar{f}_\sigma"] \arrow[rrr, bend right, "g_{\tau\sigma}"'] & K^{(n)}/K^{(n-1)} \arrow[r, "\bar{g}"] & L^{(n)}/L^{(n-1)} \arrow[r] \arrow[ur] & S^n
\end{tikzcd}$$

- For $\sigma \in \cancel{K}\, C_{n+1}(K)$, $\partial\sigma = \sum_{\tau : n\text{-cell}} [\tau;\sigma]\,\tau$

- $g_\Delta(\sigma) = \sum_{\tau : n+1 \text{ cell}} dg(g_{\tau,\sigma})\,\tau$.

- $S^n = * \cup_{\partial e_n} e_n \qquad D^n = e_n$.

~~Suppose $\varphi : K^{(n)} \to S^n$ be a cellular map~~

Want $[K; S^n] \longrightarrow H^n(K;\mathbb{Z})$

<u>Constructing</u> $\mathrm{Hom}_{\text{cellular}}(K; S^n) \longrightarrow C^n(K;\mathbb{Z})$

$\varphi : K^{(n)} \to S^n$ be cellular

$\varphi_\Delta : C^n(K^{(n)}) = C^n(K) \longrightarrow C_n(S^n)$ is given by

$$\varphi_\Delta(\tau) = \deg(\varphi_{e_n,\tau})\, e_n$$

$$\begin{tikzcd}
D^n \arrow[r, "f_\tau"] \arrow[d, "r_n"] & K^{(n)} \arrow[r, "\varphi"] \arrow[d] & (S^n)^{(n)} \arrow[r, "p_{e_n}"] \arrow[d] & S^n \arrow[d, equal] \\
S^n \arrow[r] \arrow[rrr, bend right, "\varphi_{e_n,\tau}"'] & \frac{K^{(n)}}{K^{(n-1)}} \arrow[r] & S^n \arrow[r, equal] & S^n
\end{tikzcd}$$

<u>DEFINE</u>: $c_\varphi : C_n(K) \longrightarrow \mathbb{Z}$

$$c_\varphi(\tau) = \deg(\varphi_{e_n,\tau})$$

Clearly $c_\varphi \in \cancel{C^n(K;\mathbb{Z})}\ C^n(K;\mathbb{Z})$

<u>CLAIM</u>: c_φ is a cocycle; i.e. $\delta c_\varphi = 0$

For the claim, we prove the proposition:

<u>PROPOSITION</u>

σ be an $(n+1)$ cell of K. Suppose $\varphi : K^{(n)} \to S^n$ be a map defined on the n-skeleton. Then

$$(\delta c_\varphi)(\sigma) = c_\varphi(\partial\sigma) = \cancel{\deg(\varphi \circ \partial\sigma)}. \deg(\varphi \circ f_{\partial\sigma}).$$

Moreover if φ comes from a map $K \to S^n$, then

$$\delta c_\varphi = 0$$

<u>Proof</u>: without loss we can assume that f_σ is a cellular map. [Only have to make sure $*$ goes to a pt. in $K^{(0)}$.]

~~Consider $e_{n+1} \in \mathcal{E}$~~

We construct the following diagram:

$$
\begin{tikzcd}
C_{n+1}(D^{n+1}) \arrow[r, "(f_\sigma)_\Delta"] \arrow[d, "\partial"'] & C_{n+1}(K) \arrow[d, "\partial"] & \\
C_n(D^{n+1}) & C_n(K) \arrow[r, "\varphi_\Delta"] \arrow[d, phantom, "\shortparallel"] & C_n(S^n) \\
C_n(S^n) \arrow[u, hook] \arrow[ur, "(f_{\partial\sigma})_\Delta"] & C_n(K^{(n)}) &
\end{tikzcd}
$$

Now consider e_{n+1} ~~on~~ in $C_{n+1}(D^{n+1})$ and look at its image in $C_n(S^n)$ in two ways:

(I) $(f_\sigma)_\Delta(e_{n+1}) = \sum\limits_{\substack{\tau : n+1 \text{ cell} \\ \text{in } K}} \deg(f_\sigma)_{\tau, e_{n+1}} \tau$

Let us draw the diagram computing $(f_\sigma)_{\tau, e_{n+1}}$.

$$
\begin{tikzcd}
D^{n+1} \arrow[r, "f_{e_{n+1}}"] \arrow[d, "r_{n+1}"'] & (D^{n+1})^{(n+1)} \arrow[r, "f_\sigma"] \arrow[d] & K^{(n+1)} \arrow[r, "p_\tau"] \arrow[d] & S^{n+1} \arrow[d, equal] \\
S^{n+1} \arrow[r, "id"] & \dfrac{(D^{n+1})^{(n+1)}}{(D^{n+1})^{(n)}} \arrow[r, "\bar{f}_\sigma"] \arrow[d, phantom, "\cong" rotate=90] & \dfrac{K^{(n+1)}}{K^{(n)}} \arrow[r, "\bar{p}_\tau"] & S^{n+1} \\
& S^{n+1} & &
\end{tikzcd}
$$

$D^{n+1} = D^{n+1} \cup_{f_{e_{n+1}}} S^n$
$= D^{n+1} \cup_{f_{e_{n+1}}} D^n \cup_{f_{e^n}} *$

Note if $\tau \neq \sigma$, $\bar{p}_\tau \circ \bar{f}_\sigma$ ~~is~~ is just collapsing everything to the base point. If $\tau = \sigma$, then $(f_\sigma)_{\sigma, e_{n+1}} = id$

$\Rightarrow (f_\sigma)_\Delta(e_{n+1}) = \sigma.$

$$\Rightarrow \varphi_\Delta \circ \partial \circ (f_\sigma)_\Delta (e_{n+1}) = \varphi_\Delta(\partial\sigma) = c_\varphi(\partial\sigma)e_n$$

II On the other hand,

$$\partial(e_{n+1}) = [e_{n+1}, e_n]e_n$$

But $[e_{n+1}, e_n] = \deg(p_{e_n} \circ f_{\partial e_{n+1}})$

$$\begin{tikzcd} S^n \arrow[r, "f_{\partial e_{n+1}}"', "id"] & (D^{n+1})^{(n)} \arrow[r, "id"] \arrow[rr, bend left, "p_{e_n}"] & \cancel{D^{(n+1)}}\,\dfrac{(D^{n+1})^{(n)}}{(D^{n+1})^{(n-1)}} \arrow[r, "id"] & S^n \end{tikzcd}$$

$(D^{n+1})^{(n)} = S^n$, $\quad \dfrac{(D^{n+1})^{(n)}}{(D^{n+1})^{(n-1)}} = S^n/_* \cong S^n$

$$\Rightarrow \deg(p_{e_n} \circ f_{\partial e_{n+1}}) = 1.$$

$$\Rightarrow [e_{n+1}, e_n] = 1$$

$$\Rightarrow \partial e_{n+1} = e_n$$

Note $e_n \in C_n(S^n)$

& $\varphi_\Delta((f_{\partial\sigma})_\Delta)(e_n) = (\varphi \circ f_{\partial\sigma})_\Delta(e_n)$

$$= \cancel{[e_n, e_n]}\ \deg(\varphi \circ f_{\partial\sigma})_{e_n, e_n}\ e_n$$

$$= \deg(\varphi \circ f_{\partial\sigma})\, e_n$$

$$\Rightarrow \cancel{c_\varphi(\partial\sigma)}\ \delta c_\varphi(\sigma) = c_\varphi(\partial\sigma) = \deg(\varphi \circ f_{\partial\sigma}).$$

Now, if φ comes from K, we can complete the diagram into

$$\begin{tikzcd}
C_{n+1}(D^{n+1}) \arrow[r, "(f_\sigma)_\Delta"] \arrow[d, "\text{Induced by } \partial"'] & C_{n+1}(K) \arrow[r] \arrow[d, "\partial"'] & C_{n+1}(S^n) = 0 \arrow[d, "\partial"] \\
C_n(S^n) \arrow[r, "(f_{\partial\sigma})_\Delta"] & C_n(K) \arrow[r, "\varphi_\Delta"] & C_n(S^n)
\end{tikzcd}$$

$\square$

$\Rightarrow \delta C_\varphi(\sigma) = 0 \quad \forall \sigma$

20/03/2013

Yesterday:

We defined a map

$$\mathrm{Hom}_{\text{Cellular maps}}(K, S^n) \longrightarrow C^n(K;\mathbb{Z})$$

$$\varphi \longmapsto c_\varphi \text{ where}$$

$\varphi_\Delta(\tau) = c_\varphi(\tau)\, e_n$

τ: n cell.

We also proved that $\delta c_\varphi = 0$

Moreover, we proved that if $\psi : K^{(n)} \longrightarrow S^n$ is a cellular map, then $\delta c_\psi(\sigma) = \deg(\psi \circ f_{\partial\sigma})$.

Result from a previous chapter:

[If $\varphi, \psi : K \longrightarrow S^n$ are cellular, and $\varphi \simeq \psi$ then they are homotopic via a homotopy $K \times I \longrightarrow S^n$ which takes $(K \times I)^{(n-1)}$ to the base point $*$

$$(K\times I)^{(n-1)} = (K^{(n-1)} \times \partial I) \cup (K^{(n-2)} \times I)$$

]

General proposition: Suppose we have a (homotopy)

$$F : (K \times I)^{(n)} \longrightarrow S^n$$

$$(K \times I)^{(n)} = (K^{(n-1)} \times I \cup K^{(n)} \times \partial I)$$

s.t. $F(x, 0) = \varphi_0(x)$

$F(x, 1) = \varphi_1(x)$

$x \in K^{(n)}$

[Note $K^{(n)} \times \partial I \subset (K \times I)^{(n)}$]

~~Then if~~ and if $d_F \in C^{n-1}(K,\mathbb{Z})$ is defined by

$$d_F(\tau) = c_F(\tau \times I) = \deg F_{e_n,\, \tau\times I}$$

THEN: $$\delta d_F(\sigma) = \deg(F\circ f_{\partial(\sigma\times I)}) + (-1)^{n+1}(c_{\varphi_1} - c_{\varphi_0})(\sigma)$$ σ: n cell of K

<u>Proof:</u> $$\delta d_F(\sigma) = d_F(\partial\sigma) = c_F(\partial\sigma \times I)$$

$$= c_F\big(\partial(\sigma\times I) - (-1)^n(\sigma\times\{1\}) + (-1)^n(\sigma\times\{0\})\big)$$

$$= c_F(\partial(\sigma\times I)) - (-1)^n c_F(\sigma\times\{1\}) + (-1)^n c_F(\sigma\times\{0\})$$

$$= \delta c_F(\sigma\times I) - (-1)^n \deg F_{e_n,\sigma\times\{1\}} + (-1)^n \deg F_{e_n,\sigma\times\{0\}}$$

$$\underset{\text{prev propn}}{=} \deg(F\circ f_{\partial(\sigma\times I)}) + (-1)^{n+1}\left[c_{\varphi_1}(\sigma) - c_{\varphi_0}(\sigma)\right] \qquad \square$$

$$\boxed{\begin{aligned} \because\ F|_{-\times\{1\}} &= \varphi_1 \\ F|_{-\times\{0\}} &= \varphi_0 \end{aligned}}$$

Now if F is ~~a homotopy from K to S^n~~ actually defined on $K^{(n)}\times I$,

$$\begin{tikzcd} S^n \arrow[r, "f_{\partial(\sigma\times I)}"] \arrow[d] & (K\times I)^{(n)} \arrow[r, "F"] \arrow[d, hook] & S^{(n)} \\ D^{n+1} \arrow[r, "f_{\sigma\times I}"] & K^{(n)}\times I \arrow[ur] & \end{tikzcd}$$

gives a homotop of $F\circ f_{\partial(\sigma\times I)}$ to the constant map

$$\Rightarrow \deg(F\circ f_{\partial(\sigma\times I)}) = 0$$

In particular:

If $\varphi_0 \simeq \varphi_1 : K \to S^n$ are cellular maps, then

$$c_{\varphi_1} - c_{\varphi_0} = (-1)^{n+1} \delta d_F$$

Pf: F defining the homotopy is defined on $K^{(n)} \times I$! □

Defn: $\varphi : K \to S^n$ be an arbitrary continuous map

We know that $\exists$ a cellular map $\varphi' : K \to S^n$ s.t.

$\varphi \simeq \varphi'$.

Define $\xi_\varphi = c_{\varphi'} \mod \delta(C^{n-1}(K, \mathbb{Z}))$.

This makes sense since for any other choice of a cellular map φ'', $c_{\varphi'} - c_{\varphi''} \in \delta(C^{n-1}(K, \mathbb{Z}))$!

This gives us a map

$$[K, S^n] \longrightarrow H^n(K, \mathbb{Z}) = \frac{\operatorname{Ker}(C^n(K, \mathbb{Z}) \xrightarrow{\delta} C^{n+1}(K, \mathbb{Z}))}{\operatorname{Im}(C^{n-1}(K, \mathbb{Z}) \xrightarrow{\delta} C^n(K, \mathbb{Z}))}$$

$$\varphi \longmapsto \xi_\varphi .$$

Thm: $\varphi_0, \varphi_1 : K \to S^n$

(1) $\varphi_0 \simeq \varphi_1 \iff \xi_{\varphi_0} = \xi_{\varphi_1}$ (Injectivity)

(2) $\forall \xi \in H^n(K, \mathbb{Z})\ \exists\ \varphi : K \to S^n$ s.t. $\xi_\varphi = \xi$ (Surjectivity)

We first prove ②:

$\xi \in H^n(K,\mathbb{Z})$. $\Rightarrow$ choose any cocycle $c \in C^n(K,\mathbb{Z})$ $(\delta c = 0)$ s.t. ξ is the class of c.

Consider the map

$$\begin{tikzcd} K = K^{(n)} \arrow[r] \arrow[rrr, bend right, "\varphi"'] & \dfrac{K^{(n)}}{K^{(n-1)}} \arrow[r, "p_\tau"] & \bigvee\limits_{\tau: n\,\text{cell}} S^n_\tau \arrow[r, "\vee \alpha_\tau"] & S^n \end{tikzcd}$$

where α_τ is a map $\alpha_\tau : S_\tau \to S_\tau$ of degree $c(\tau)$.

$\Rightarrow c_\varphi(\tau) = \deg \varphi_{\tau} = \deg \alpha_\tau = c(\tau) \ \forall \tau$

$\Rightarrow \xi_\varphi = \xi$ $\square$

Proof of

① We already know $\varphi_0 \simeq \varphi_1 \Rightarrow \xi_{\varphi_0} = \xi_{\varphi_1}$

Now suppose $\xi_{\varphi_0} = \xi_{\varphi_1}$. By definition, without loss, we can assume φ_0 and φ_1 are cellular.

~~Let c_{φ_0}~~ Let c_{φ_0} and c_{φ_1} be cocycles representing ξ_{φ_0} & ξ_{φ_1}

$\xi_{\varphi_0} = \xi_{\varphi_1} \Rightarrow c_{\varphi_0} - c_{\varphi_1} \in \delta(C^{n-1}(K)) \Rightarrow (-1)^{n+1}(c_{\varphi_0} - c_{\varphi_1}) = \delta d$. for some $d \in C^{n-1}(K) = \mathrm{Hom}(C_{n-1}(K),\mathbb{Z})$

This helps us to define

$$F : (K\times I)^{(n)} = \big(\underbrace{K^{(n)} \times \partial I}_{= K\times\partial I}\big) \cup (K^{(n-1)} \times I) \longrightarrow S^n \quad \text{by}$$

ⓐ F takes $(K\times I)^{(n-1)}$ to the base pt. $*$

ⓑ $F|_{K\times\{0\}} = \varphi_0$, $F|_{K\times\{1\}} = \varphi_1$

ⓒ For τ an $(n-1)$ cell of K, $\deg F_{e_n;\,\tau\times I} = d(\tau)$

Claim:

For such an F,

$$d_F = d$$

This is because, for an $(n-1)$ cell τ,

$$d_F(\tau) = c_F(\tau \times I) = \deg F_{e_n, \tau\times I} = d(\tau) \text{ by construction}$$

$$\Rightarrow \delta d_F(\sigma) = \delta d(\sigma) \quad \forall \sigma$$

But $\delta d_F(\sigma) = \deg(F \circ f_{\partial(\sigma\times I)}) + (-1)^{n+1}(c_{\varphi_1} - c_{\varphi_0})(\sigma)$

and $\delta d(\sigma) = (-1)^{n+1}(c_{\varphi_1} - c_{\varphi_0})(\sigma)$ by choice of d

$$\Rightarrow \deg(F \circ f_{\partial(\sigma\times I)}) = 0$$

$\Rightarrow F$ extends to $\sigma \times I$

~~By induction~~ Since this is true for any n-cell σ,

F extends to $\underset{\underset{K\times I}{\parallel}}{(K\times I)^{(n+1)}} = K^{(n)} \times I$. $\quad (\because K^{(n+1)} = K^{(n)})$

Thus F gives a homotopy between φ_0 and φ_1.

<u>Exercise</u> (Reading) For any CW complex K,

$$[K, S^1] \cong H^1(K, \mathbb{Z}).$$