Midterm solutions

① (i) This is an application of the Universal Coefficient Theorem.
$X = \mathbb{K}^2$, $G = \mathbb{Z}$.

$$0 \to \mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}) \to H^n(X;\mathbb{Z}) \to \mathrm{Hom}(H_n(X), \mathbb{Z}) \to 0$$

which splits (non-functorially).

$\Rightarrow$ As groups, $H^n(X;\mathbb{Z}) \cong \mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}) \oplus \mathrm{Hom}(H_n(X), \mathbb{Z})$

Now for $n \geq 3$ both $H_{n-1}(X)$ and $H_n(X)$ are zero

$\therefore$ The only non-zero $H^n(X,\mathbb{Z})$ ~~will~~ might be ~~for n=0,1 or 2~~ for $n \in \{0,1,2\}$.

$$H^0(X;\mathbb{Z}) \cong 0 \oplus \cancel{\mathrm{Ext}}\ \mathrm{Hom}(H_0(X), \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}.$$

$$\begin{aligned}
H^1(X;\mathbb{Z}) &\cong \mathrm{Ext}^1(H_0(X),\mathbb{Z}) \oplus \mathrm{Hom}(H_1(X),\mathbb{Z}) \\
&\cong \mathrm{Ext}^1(\mathbb{Z},\mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\
&\cong 0 \oplus \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\
&\cong 0 \oplus \mathbb{Z} \oplus 0 \\
&\cong \mathbb{Z}.
\end{aligned}$$

$$\begin{aligned}
H^2(X,\mathbb{Z}) &\cong \mathrm{Ext}^1(H_1(X),\mathbb{Z}) \oplus \mathrm{Hom}(H_2(X),\mathbb{Z}) \\
&\cong \mathrm{Ext}^1\left(\mathbb{Z} \oplus \tfrac{\mathbb{Z}}{2\mathbb{Z}}, \mathbb{Z}\right) \oplus \mathrm{Hom}(0,\mathbb{Z}) \\
&\cong \mathrm{Ext}^1(\mathbb{Z},\mathbb{Z}) \oplus \mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \oplus 0 \\
&\cong 0 \oplus \mathbb{Z}/2\mathbb{Z} \oplus 0 \\
&\cong \mathbb{Z}/2\mathbb{Z}
\end{aligned}$$

$$0 \to \underset{1}{\mathbb{Z}} \xrightarrow{\times 2} \underset{0}{\mathbb{Z}} \to 0$$

$$0 \to \underset{0}{\mathrm{Hom}(\mathbb{Z},\mathbb{Z})} \xrightarrow[=(\times 2)]{\circ(\times 2)} \underset{1}{\mathrm{Hom}(\mathbb{Z},\mathbb{Z})} \to 0$$

To compute $H_i(X, \mathbb{Z}/p\mathbb{Z})$, we use the UCT

$$0 \to H_n(X) \otimes \mathbb{Z}/p \to H_n(X; \mathbb{Z}/p) \to \mathrm{Tor}_1(H_{n-1}(X), \mathbb{Z}/p) \to 0$$

which also splits to give

$$H_n(X; \mathbb{Z}/p) \cong (H_n(X) \otimes \mathbb{Z}/p) \oplus \mathrm{Tor}_1(H_{n-1}(X), \mathbb{Z}/p)$$

As before $H_n(X, \mathbb{Z}/p) = 0$ for $n < 0$ & $n > 2$

Now

$$H_0(X) \otimes \mathbb{Z}/p \cong \mathbb{Z} \otimes \mathbb{Z}/p \cong \mathbb{Z}/p$$

$$H_1(X) \otimes \mathbb{Z}/p \cong (\mathbb{Z} \oplus \mathbb{Z}/2) \otimes \mathbb{Z}/p \cong (\mathbb{Z} \otimes \mathbb{Z}/p) \oplus (\mathbb{Z}/2 \otimes \mathbb{Z}/p)$$

$$\cong \begin{cases} \mathbb{Z}/p \oplus 0 \cong \mathbb{Z}/p & \text{if } p \neq 2 \text{ prime} \\ \mathbb{Z}/p \oplus \mathbb{Z}/2 & \text{if } p = 2 \end{cases}$$

$$H_2(X) \otimes \mathbb{Z}/p \cong 0 \otimes \mathbb{Z}/p = 0$$

and

$$\mathrm{Tor}_1(H_{-1}(X), \mathbb{Z}/p) = 0$$

$$\mathrm{Tor}_1(H_0(X), \mathbb{Z}/p) \cong \mathrm{Tor}_1(\mathbb{Z}, \mathbb{Z}/p) \cong 0 \quad (\mathbb{Z} \text{ is projective})$$

$$\mathrm{Tor}_1(H_1(X), \mathbb{Z}/p) \cong \mathrm{Tor}_1(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/p)$$

$$\cong \mathrm{Tor}_1(\mathbb{Z}, \mathbb{Z}/p) \oplus \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/p)$$

$$\cong \cancel{0 \oplus} \begin{cases} 0 \oplus \mathbb{Z}/2 & \text{when } p = 2 \\ 0 \oplus 0 & \text{if } p \neq 2 \end{cases}$$

$$\cong \begin{cases} \mathbb{Z}/2 & \text{when } p = 2 \\ 0 & \text{o.w.} \end{cases}$$

(p - prime)

Putting together

$$H_0(X, \mathbb{Z}/p) \cong (H_0(X) \otimes \mathbb{Z}/p) \oplus \mathrm{Tor}_1(H_{-1}(X), \mathbb{Z}/p)$$

$$\cong \mathbb{Z}/p$$

$$H_1(X, \mathbb{Z}/p) \cong H_1(X) \otimes \mathbb{Z}/p \oplus \mathrm{Tor}_1(H_0(X), \mathbb{Z}/p)$$

$$\cong \begin{cases} \mathbb{Z}/p \oplus \mathbb{Z}/2 \oplus 0 \cong \mathbb{Z}/p \oplus \mathbb{Z}/2 & \text{for } p=2 \\ \mathbb{Z}/p \oplus 0 \oplus 0 \cong \mathbb{Z}/p & \text{for } p \neq 2 \end{cases}$$

$$H_2(X, \mathbb{Z}/p) \cong H_2(X) \otimes \mathbb{Z}/p \oplus \mathrm{Tor}_1(H_1(X), \mathbb{Z}/p)$$

$$\cong \begin{cases} 0 \oplus \mathbb{Z}/2 \cong \mathbb{Z}/2 & \text{when } p=2 \\ 0 \oplus 0 \cong 0 & \text{otherwise} \end{cases}$$

(p prime)

1(iii) In this case, the version of UCT we use will be

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}/p) \longrightarrow H^n(X; \mathbb{Z}/p) \longrightarrow \mathrm{Hom}(H_n(X), \mathbb{Z}/p) \longrightarrow 0$$

which splits (but not naturally)

$$\cong H^n(X, \mathbb{Z}/p) \cong \mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}/p) \oplus \mathrm{Hom}(H_n(X), \mathbb{Z}/p)$$

Now $\mathrm{Ext}^1(\overset{=0}{H_{-1}(X)}, \mathbb{Z}/p) = 0$

$\mathrm{Ext}^1(H_0(X), \mathbb{Z}/p) \cong \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}/p) = 0$ ($\mathbb{Z}$ is projective)

$\mathrm{Ext}^1(H_1(X), \mathbb{Z}/p) \cong \mathrm{Ext}^1(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$

$\cong \mathrm{Ext}^1(\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) \oplus \mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$

$\cancel{\cong \oplus}$

$$\cong \begin{cases} 0 \oplus \mathbb{Z}/2\mathbb{Z} & \text{when } p=2 \\ 0 \oplus 0 & \text{when } p \neq 2 \end{cases}$$

$\Rightarrow$

$H^0(X, \mathbb{Z}/p) \cong \mathrm{Ext}^1(H_{-1}(X), \mathbb{Z}/p) \oplus \mathrm{Hom}(H_0(X), \mathbb{Z}/p)$

$\cong 0 \oplus \mathbb{Z}/p \cong \mathbb{Z}/p$

$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0$

projres of $\mathbb{Z}/2$

$0 \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/p) \xrightarrow{\times 2} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/p) \to 0$

$\Rightarrow \mathrm{Ext}^1 = \ker(\times 2)$

$$= \begin{cases} 0 & \text{if } p \neq 2 \\ \mathbb{Z}/2 & \text{if } p = 2 \end{cases}$$

$H^1(X, \mathbb{Z}/p) \cong \mathrm{Ext}^1(H_0(X), \mathbb{Z}/p) \oplus \mathrm{Hom}(H_1(X), \mathbb{Z}/p)$

$\cong 0 \oplus (\mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/p))$

$= 0 \oplus \mathbb{Z}/p \oplus \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/p)$

$$= \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } p=2 \\ \mathbb{Z}/p \oplus 0 \cong \mathbb{Z}/p & \text{if } p \neq 2 \text{, prime} \end{cases}$$

$H^2(X, \mathbb{Z}/p) = \mathrm{Ext}^1(H_1(X), \mathbb{Z}/p) \oplus \mathrm{Hom}(H_2(X), \mathbb{Z}/p)$

$= \mathrm{Ext}^1(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/p) \oplus 0$

$$= \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } p=2 \\ 0 & \text{o.w.} \end{cases}$$

2 The aim is to use the version of UCT giving the following short exact sequence

$$0 \to H_n(C_*) \otimes G \to H_n(C_* \otimes G) \to \cancel{H_{n-1}(C_*)}$$
$$\mathrm{Tor}_1(H_{n-1}(C_*), G) \to 0$$

~~Consider~~

Take $C_{-k} = \mathrm{Hom}(C_k(X,A), \mathbb{Z}) = C^k(X,A;\mathbb{Z})$

$\partial : C_{-k} \to C_{-k-1}$ is taken to be $\delta : C^k(X,A) \to C^{k+1}(X,A)$

Note $H_{-n}(C_*) = H^n(C^*(X,A;\mathbb{Z})) = H^n(X,A;\mathbb{Z})$

Now ~~C_{-n}~~ $C_{-n} \otimes G = C^n(X,A;\mathbb{Z}) \otimes G$
$= \mathrm{Hom}(C_n(X,A),\mathbb{Z}) \otimes G$

<u>Claim:</u> If G is finitely generated, $\mathrm{Hom}(C,\mathbb{Z}) \otimes G \cong \mathrm{Hom}(C,G)$ where C is a free abelian group

<u>Proof:</u> $\because G$ is finitely generated, there exists a surjective map $\mathbb{Z}^g \xrightarrow{\varphi} G$ for some g (= # of a set of generators for G).
Let $K = \ker\varphi$. Being a subgroup of a free abelian group, $K \cong \mathbb{Z}^k$ for some integer $k \geq 0$.

$$0 \to \mathbb{Z}^k \to \mathbb{Z}^g \to G \to 0$$

is a s.e.s.

Now we have the l.e.s.

$$\mathrm{Tor}_1(\mathrm{Hom}(C,\mathbb{Z}),G) \to \mathrm{Hom}(C,\mathbb{Z}) \otimes \mathbb{Z}^k \to \mathrm{Hom}(C,\mathbb{Z}) \otimes \mathbb{Z}^g \to \mathrm{Hom}(C,\mathbb{Z}) \otimes G$$
$$\cancel{\to \mathrm{Ext}^1(}$$
$$\cancel{\to \mathrm{Tor}_1(} \to 0$$

$\because$ C is free, so ~~is~~ $\mathrm{Hom}(C,\mathbb{Z})$

$\Rightarrow \mathrm{Tor}_1(\mathrm{Hom}(C,\mathbb{Z}), G) = 0$

Furthermore, $\mathrm{Hom}(C,\mathbb{Z}) \otimes \mathbb{Z}^l \cong \bigoplus_{i=1}^{l} \mathrm{Hom}(C,\mathbb{Z})$

$\cong \mathrm{Hom}(C, \bigoplus_{i=1}^{l} \mathbb{Z}) = \mathrm{Hom}(C,\mathbb{Z}^l)$

for any integer $l \geq 0$; in particular for $l = k, g$.

$\Rightarrow 0 \to \mathrm{Hom}(C,\mathbb{Z}^k) \to \mathrm{Hom}(C,\mathbb{Z}^g) \to \mathrm{Hom}(C,\mathbb{Z}) \otimes G \to 0$ is exact ---------- (A)

On the other hand, applying $\mathrm{Ext}^i(C,-)$ to the s.e.s.

$$0 \to \mathbb{Z}^k \to \mathbb{Z}^g \to G \to 0$$

we get

$0 \to \mathrm{Hom}(C,\mathbb{Z}^k) \to \mathrm{Hom}(C,\mathbb{Z}^g) \to \mathrm{Hom}(C,G) \to \mathrm{Ext}^1(C,\mathbb{Z}^k) \to \ldots$

Since C, is free, it is projective $\Rightarrow \mathrm{Ext}^1(C,\mathbb{Z}^k) = 0$

$\Rightarrow 0 \to \mathrm{Hom}(C,\mathbb{Z}^k) \to \mathrm{Hom}(C,\mathbb{Z}^g) \to \mathrm{Hom}(C,G) \to 0$ is exact (B)

Comparing A & B, we get

$$\mathrm{Hom}(C,\mathbb{Z}) \otimes G \cong \mathrm{Hom}(C,G)$$

for any finitely generated ~~group~~ abelian group G.

Now $C_* = C^{-*}(X;A;\mathbb{Z})$

& $C_* \otimes G = C^{-*}(X,A;G)$

$\Rightarrow$ ~~H_k~~ $H_{-k}(C_*) = H^k(C^*(X,A;\mathbb{Z})) = H^k(X;A;\mathbb{Z})$

& $H_{-k}(C_* \otimes G) = H^k(C^*(X,A;G)) = H^k(X,A;G)$

$\therefore$ UCT $\Rightarrow$

$0 \to H^n(X,A;\mathbb{Z}) \otimes G \to H^n(X,A;G) \to \mathrm{Tor}_1(H^{n-1}(X,A;\mathbb{Z}), G) \to 0$

is exact & splits naturally in G.

3 (i) ~~$Ext^1(M,N)=0$ $\forall$ R modules M will~~ i.e.

Consider any surjective map of R-modules

$$\alpha : Q \to Q''$$

Suppose Q' is the kernel giving us an s.e.s. $0 \to Q' \to Q \to Q'' \to 0$

To see that M is projective we have to prove that

for any map $\psi : M \to Q''$, $\exists$ some map $\tilde{\psi} : M \to Q$

s.t. $\psi = \alpha \circ \tilde{\psi}$

$$\begin{tikzcd} & M \arrow[dl, dashed, "\text{To show } \exists \tilde{\psi}"'] \arrow[d, "\psi"] & \\ Q \arrow[r, "\alpha"'] & Q'' \arrow[r] & 0 \end{tikzcd}$$

To see that look at the l.e.s

$$0 \to Hom(M, Q') \to Hom(M, Q) \xrightarrow{\alpha \circ} Hom(M, Q'') \to Ext^1(M, Q') \to \cdots$$

$$\underset{\psi}{\in}$$

But $Ext^1(M, Q') = 0$ by assumption

Therefore, $Hom(M,Q) \to Hom(M,Q')$ is surjective

In particular, $\exists\, \tilde{\psi}$ s.t. $(\alpha \circ)\tilde{\psi} = \psi$

$\Rightarrow \alpha \circ \tilde{\psi} = \psi$ $\square$

3 (ii) The answer is no. We shall prove that every projective module is torsion free. I'll give 2 proofs.

Proof 1 Suppose $\exists\, p \in P$ s.t. $np = 0$. n is the minimal such positive no.

Suppose ~~$C \subseteq P$ be the subgroup generated~~ by p.

~~Let $Q = P/C$~~

__Claim:__ $\mathrm{Tor}_1(P, \mathbb{Z}/n\mathbb{Z}) \neq 0$

Note that this would contradict the fact that P is projective.

Consider the projective resolution

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to 0 \quad \text{for } \mathbb{Z}/n\mathbb{Z}$$

$\Rightarrow$ Tor_1 will be computed by

$$\begin{array}{ccccccc} 0 \to & P & \xrightarrow{\times n} & P & \to 0 \\ & \wr\vert & & \wr\vert & \\ & \mathbb{Z}\otimes P & & \mathbb{Z}\otimes P & \\ & 1 & & 0 & \end{array}$$

$\mathrm{Tor}_1(P, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Ker}(P \xrightarrow{\times n} P)$

Definitely $\phi \in$ ~~[illegible]~~ $\mathrm{Ker}(P \xrightarrow{\times n} P)$

$$\Rightarrow \mathrm{Tor}(P, \mathbb{Z}/n\mathbb{Z}) \neq 0$$

__Proof II:__ We directly prove that P is a direct summand of a free group.

Considering ~~gener~~ the free group F generated by a set of generators of P, we get a surjection

$$F \xrightarrow{\alpha} P \xrightarrow{\pi} 0$$

Let K be the ~~ker~~ kernel. We get the short exact seq.

$$\begin{tikzcd} & & & P \arrow[d, "="] \arrow[dl, dotted, "i"'] & \\ 0 \arrow[r] & K \arrow[r, hook, "\alpha"'] & F \arrow[r, "\pi"'] & P \arrow[r] & 0 \end{tikzcd}$$

Since P, is projective $\exists\, i : P \to F$ s.t. $\pi \circ i = id_P$.

$\Rightarrow F \cong K \oplus i(P) \Rightarrow$ $i(P)$ is torsion free (being a subgroup of a free group)

$\Rightarrow$ P is torsion free

Let the maps in the given s.e.s. be $0 \to N \xrightarrow{i} X \xrightarrow{p} M \to 0$

4 (i) Consider the long exact sequence

$$0 \to \mathrm{Hom}(M,N) \xrightarrow[(i\circ)]{} \mathrm{Hom}(M,X) \xrightarrow[(p\circ)]{} \mathrm{Hom}(M,M) \xrightarrow[\delta]{} \cancel{\mathrm{Hom}}\ \underset{\overset{\|}{0}\ \text{given}}{\mathrm{Ext}^1(M,N)}$$

$\Rightarrow$ $(p\circ)$ is surjective

$\Rightarrow \exists\ s \in \mathrm{Hom}(M,X)$ s.t. $(p\circ)s = p\circ s = id_M \in \mathrm{Hom}(M,M)$

$\Rightarrow$ The s.e.s.
$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i"'] & X \arrow[r, "p"'] & M \arrow[r] \arrow[l, bend right, "s"'] & 0 \end{tikzcd}$$
splits.

~~& hence X~~

4 (ii) Consider an extension ~~$0 \to N \to$~~

$$0 \to \mathbb{Z}/p\mathbb{Z} \to E \to \mathbb{Z}/p\mathbb{Z} \to 0$$

$\Rightarrow$ The number of elements in $E = p^2$

Being a $\mathbb{Z}$-module (or abelian group of order p^2),

$E \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$ or $E \cong \mathbb{Z}/p^2$

Now ~~let~~ a surjective map $\mathbb{Z}/p^2 \to \mathbb{Z}/p$ is given

by $\varphi: 1 \mapsto a$ where $a \in (\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}$ [$1 \mapsto 0$ is not surjective]

This gives us $(p-1)\cdot$ short exact sequences

~~$\mathbb{Z}/p\mathbb{Z}$~~

$$E(a): \quad 0 \to \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \xrightarrow[1 \mapsto a]{\varphi_a} \mathbb{Z}/p\mathbb{Z} \to 0.$$

Thus there are possibly $(p-1)$ extensions involving $\mathbb{Z}/p^2\mathbb{Z}$ and we shall show there is only one extension involving $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ upto equivalence. Once we show that the $E(a)$'s are non-equivalent, $E(a)$'s will give us exactly $p-1$ non-equivalent extensions, giving us a total of p extensions.

Choose a, b. Suppose $E(a)$ & $E(b)$ are equivalent.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p \arrow[r] \arrow[d, equal, "\psi'"'] & \mathbb{Z}/p^2 \arrow[r, "\varphi_a"] \arrow[d, "\psi"] & \mathbb{Z}/p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/p \arrow[r] & \mathbb{Z}/p^2 \arrow[r, "\varphi_b"] & \mathbb{Z}/p \arrow[r] & 0
\end{tikzcd}$$

Note ψ is determined by $\psi(1)$, say $\psi(1) = k$

$\Rightarrow \varphi_a(1) = \varphi_b \circ \psi(1)$

$\Rightarrow a \equiv k b \mod p$

Note that $\ker \varphi_a = \ker \varphi_b = \{0, p, 2p, \dots, (p-1)p\} \cong \mathbb{Z}/p$

& $\psi(p) = kp \Rightarrow \psi'(1) = k$

$\because \psi'$ is the identity, this implies $k \equiv 1 \mod p$

$\Rightarrow a \equiv b \mod p$ $\square$.

5. ω is a 1 form on $\mathbb{R}^2\setminus\{0\}$ and hence can be written in the form

$$\omega = a(r,\theta)\,dr + b(r,\theta)\,d\theta$$

where $a(r,\theta)$ and $b(r,\theta)$ are 2π periodic in θ.

$$d\omega = 0 \Rightarrow \left(-\frac{\partial a}{\partial \theta}\right) dr\wedge d\theta + \frac{\partial b}{\partial r}\, dr\wedge d\theta = 0$$

$$\Rightarrow \frac{\partial a}{\partial \theta} = \frac{\partial b}{\partial r} \qquad \text{-------------- (A)}$$

Consider ~~any~~ two circles C_u and C_v

$$C_t = \{(r,\theta) \in \mathbb{R}^2\setminus\{0\} \mid r = t\} = \{(x,y) \mid x^2+y^2 = r^2\}$$

let $u < v$ and let A be the annulus

$$\{(x,y) \mid u^2 \leq x^2+y^2 \leq v^2\}$$

Then $\displaystyle\int_{\partial A} \omega = \int_A d\omega = 0 \quad (\because d\omega = 0)$

$$\Rightarrow \int_{C_u} \omega = \int_{C_v} \omega$$

But u, v were arbitrary.

Define $\displaystyle \lambda = \int_{C_u} \omega$ for any u

Now, $\displaystyle\int_{C_u} \omega = \int_0^{2\pi} \varphi_u^* \omega = \int_0^{2\pi} b(u,\theta)\,d\theta$

$\varphi_u : [0,2\pi] \hookrightarrow \mathbb{R}^2\setminus\{0\}$

~~$\theta \mapsto (\cos\theta, \sin\theta)$~~

$\theta \mapsto (u\cos\theta, u\sin\theta)$

$\theta \mapsto (u,\theta)$

12

Consider ~~$\omega - d\theta$~~ $\omega - \lambda\, d\theta$

$$= a(r,\theta)\, dr + (b(r,\theta) - \lambda)\, d\theta$$

Let $$g(r,\theta) = \int_0^\theta (b(r,\varphi) - \lambda)\, d\varphi$$

Note g is 2π periodic by the choice of λ and hence

~~Now dg~~ defines a map on $\mathbb{R}^2 \setminus \{0\}$

$\Rightarrow$

Also $$\frac{\partial g}{\partial r} = \int_0^\theta \frac{\partial}{\partial r} (b(r,\varphi) - \lambda)\, d\varphi$$

$$= \int_0^\theta \frac{\partial b}{\partial r}(r,\varphi)\, d\varphi$$

$$= \int_0^\theta \frac{\partial a}{\partial \theta}(r,\varphi)\, d\varphi$$

$$= a(r,\theta)$$

$$\frac{\partial g}{\partial \theta} = b(r,\theta) - \lambda$$

$$\Rightarrow dg = \omega - \lambda\, d\theta$$

$\Rightarrow \omega = \lambda\, d\theta + dg$ as was to be proved $\square$

(6) First part

Consider the manifold with boundary

$$N = M_1 \setminus (\operatorname{Int} M_2)$$

Then $\partial N = \partial M_1 - \partial M_2$

$$\therefore \int_{\partial N} \omega = \int_N d\omega = 0 \quad (\because d\omega = 0)$$

$$\Rightarrow \int_{\partial M_1} \omega = \oint_{\partial M_2} \omega$$

2nd part:

Let $U_N = S^2 \setminus (-1,0,0)$

$U_S = S^2 \setminus (1,0,0)$

Define the maps j_N, j_S, i_N, i_S to be inclusions as in the diagram

$$\begin{tikzcd}
 & U_N \arrow[dr, hook, "i_N"] & \\
U_{NS} = U_N \cap U_S \arrow[ur, hook, "j_N"] \arrow[rr, hook, "k"] \arrow[dr, hook, "j_S"'] & & S^2 \\
 & U_S \arrow[ur, hook, "i_S"'] &
\end{tikzcd}$$

Define $\omega_N = i_N^* \omega$; $\omega_S = i_S^* \omega$; $\omega_{NS} = j_N^* \omega_N = j_S^* \omega_S = k^* \omega$

Note that U_N is diffeomorphic to $\mathbb{R}^2$ by the stereographic projection from $(-1,0,0)$

Similarly, U_S is diffeomorphic to $\mathbb{R}^2$

$\Rightarrow$ $\omega_N = d\theta_N$ for some function (0-form) $\theta_N : U_N \to \mathbb{R}$

& $\omega_S = d\theta_S$ " " " $\theta_S : U_S \to \mathbb{R}$

Now $j_N^* \omega_N - j_S^* \omega_S = \omega_{NS} - \omega_{NS} = 0$

$\Rightarrow j_N^* d(\theta_N) - j_S^* d(\theta_S) = 0$

$\Rightarrow d(j_N^* \theta_N - j_S^* \theta_S) = 0$

$\Rightarrow d(\theta_N|_{U_{NS}} - \theta_S|_{U_{NS}}) = 0$

$\Rightarrow \theta_N|_{U_{NS}} - \theta_S|_{U_{NS}} = C$ for some constant C ($\because U_{NS}$ is connected)

$\Rightarrow \theta_N|_{U_{NS}} = \theta_S|_{U_{NS}} + C$

Define $f(x) = \begin{cases} \theta_N(x) & \text{for } x \in U_N \\ \theta_S(x) + C & \text{for } x \in U_S \end{cases}$

which makes sense as $\theta_N|_{U_{NS}} = (\theta_S + C)|_{U_{NS}}$

~~Further more~~,

Furthermore, $df|_{U_N} = d\theta_N|_{U_N} = \omega|_{U_N}$

& $df|_{U_S} = d(\theta_S + C)|_{U_S} = \omega|_{U_S}$

$\Rightarrow df = \omega$ on S^2

$\Rightarrow \omega$ is exact.

$\square$