Mid Sem

08 March 2013

Answer **any five** of the following the questions. Try to be as precise as possible.

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1. Given that the Klein bottle $\mathbb{K}^2$ has homologies

$$H_0(\mathbb{K}^2) \cong \mathbb{Z}, \qquad H_1(\mathbb{K}^2) \cong \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$$

and $H_i(\mathbb{K}^2) = 0 \ \forall\ i \neq 0, 1$,

(i) compute $H^i(\mathbb{K}^2, \mathbb{Z})$ for all i; (*6 points.*)
(ii) compute $H_i(\mathbb{K}^2, \mathbb{Z}/p\mathbb{Z})$ for all i, and for all p; (*7 points.*)
(iii) compute $H^i(\mathbb{K}^2, \mathbb{Z}/p\mathbb{Z})$ for all i and for all p. (*7 points.*)

2. If G is finitely generated show that there is a natural exact sequence

$$0 \to H^n(X, A; \mathbb{Z}) \otimes G \to H^n(X, A; G) \to \mathrm{Tor}_1\left(H^{n+1}(X, A; \mathbb{Z}),\, G\right) \to 0$$

which splits naturally in G.

3. This question has two parts:

(i) Let R be a commutative ring with 1. If $\mathrm{Ext}^1(M, N) = 0$ for all R-modules N show that M is projective. (*10 points.*)
(ii) Can a projective module P over $\mathbb{Z}$ have an element $p \in P$ which is torsion? (p is torsion means that there exists $n \in \mathbb{N}$ such that $np = 0$.) (*10 points.*)

4. This question also has two parts. Suppose M and N are R modules, where R is a commutative ring with 1.

(i) Show that if $\mathrm{Ext}^1(M, N) = 0$, then every short exact sequence of the form

$$0 \to N \to X \to M \to 0$$

is split. (*6 points.*)
(ii) Consider the definition.
Definition. Two extensions $0 \to N \to X \to M \to 0$ and $0 \to N \to Y \to M \to 0$ are said to be equivalent if there is

an isomorphism $\varphi : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & {\cong}\downarrow{\varphi} & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Prove that there are exactly p non-equivalent extension of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$. (*14 points.*)

5. Let ω be a 1 form on $\mathbb{R}^2 \setminus \{0\}$ such that $d\omega = 0$. Prove that $\omega = \lambda d\theta + dg$ for some $\lambda \in \mathbb{R}$ and for some $g : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, and $d\theta$ is as defined in class. (*20 points.*)

6. If $M_1 \subset \mathbb{R}^n$ is an n dimensional manifold with boundary and if $M_2 \subset M_1 \setminus \partial M_1$ is another n dimensional manifold with boundary; and suppose M_1 and M_2 are compact. Then show that

$$\int_{\partial M_1} \omega = \int_{\partial M_2} \omega$$

where ω is an $n-1$ form on M_1 and ∂M_1 and ∂M_2 have orientation induced by the usual orientation on M_1 and M_2. (*10 points.*)

Using the fact that every closed differential 1-form on $\mathbb{R}^2$ is exact, prove that every close differential 1-form on the sphere S^2 is also exact. (*10 points.*)