Topology - II (End Sem)

25 April 2013

Answer the following the questions. Try to be as precise as possible. Each question carries 20 points.

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1. Throughout this problem X is a compact, orientable manifold such that all the homology groups $H_i(X;\mathbb{Z})$ are finitely generated abelian groups.

First a few definitions. The *rank of* $H_i(X,\mathbb{Z})$ is defined to be the rank of the free part of $H_i(X;\mathbb{Z})$. For example, if $H_i(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then rank of $H_i(X;\mathbb{Z})$ is 1.

The *i-th Betti number* b_i of X is defined to be the rank of $H_i(X;\mathbb{Z})$.

Now define the *Poincaré polynomial* to be $P_X(t) = \sum_{i=0}^{\infty} b_i(X)t^i$.

Suppose all the spaces we consider, have the additional property that all the homology groups are finitely generated and free.

(i) Prove that $b_i(X) = \dim_K H_i(X;K) = \dim_K H^i(X;K)$ where K is a field (of characteristic 0) and the $\dim_K$ is the dimension as a vector space.

(ii) Prove that $P_{X_1\times X_2}(t) = P_{X_1}(t)P_{X_2}(t)$.

(iii) Prove that if $\deg P_X = n$ then $P_X(t) = t^n P_X(1/t)$.

Proof. (i) Using Universal Coefficient Theorem and that the fact that free, finitely generated groups are projective, one gets,

$$H^n(X;K) = \hom(H_n(X), K) \cong K^{\operatorname{rank} H_n(X)}; \text{ and,}$$
$$H_n(X;K) = H_n(X) \otimes K = K^{\operatorname{rank} H_n(X)}.$$

This proves the first statement.

(ii) This would follow from the Künneth formula as follows. Since the homologies are finitely generated and free, the $\operatorname{Tor}_1(H_*(X), H_*(X))$ is zero, and hence,

$$H_n(X_1 \times X_2) \cong \bigoplus_{p+q=n} H_p(X_1) \otimes H_q(X_2).$$

Therefore,

$$\begin{aligned}\operatorname{rank} H_n(X_1 \times X_2) &= \sum_{p=0}^{n} \operatorname{rank} H_p(X_1) \operatorname{rank} H_{n-p}(X_2) \\ &= \sum_{p=0}^{n} b_p(X_1) b_{n-p}(X_2)\end{aligned}$$

Thus we compute

$$\begin{aligned}P_{X_1\times X_2}(t) &= \sum_{i=0}^{\infty}\operatorname{rank}(H_n(X_1\times X_2))t^n = \sum_{i=0}^{\infty}\left(\sum_{p=0}^{n} b_p(X_1)b_{n-p}(X_2)\right)t^n\\ &= \left(\sum_{p=0}^{\infty} b_p(X_1)t^p\right)\left(\sum_{q=0}^{\infty} b_q(X_2)t^q\right)\\ &= P_{X_1}(t)P_{X_2}(t).\end{aligned}$$

(iii) Now we use the fact that X is a compact orientable manifold. Let $n = \dim X$. Therefore, Poincaré duality holds. Thus,

$$\begin{aligned}b_k(X) &= \operatorname{rank}(H_k(X)) = \dim_{\mathbb{R}} H_k(X;\mathbb{R})\\ &= \dim_{\mathbb{R}} H^{n-k}(X;\mathbb{R}) = \operatorname{rank}(H_{n-k}(X))\\ &= b_{n-k}(X)\end{aligned}$$

using the first part of this problem. Therefore,

$$\begin{aligned}t^n P_X(1/t) &= t^n\sum_{k=0}^{n} b_k(X)t^{-k} = \sum_{k=0}^{n} b_{n-k}(X)t^{n-k}\\ &= \sum_{l=0}^{n} b_l(X)t^l \quad \text{where } l = n-k\\ &= P_X(t).\end{aligned}$$

□

2. (i) Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a map and ω is a *closed* 1-form on $\mathbb{R}^2$. Prove that

$$\int_{S^1}\gamma^*\omega = 0.$$

(ii) Consider the inclusion $\iota : S^1 \hookrightarrow \mathbb{R}^2$. Let $\omega = (x-y^3)dx + x^3dy$. Compute

$$\int_{S^1}\iota^*\omega.$$

Proof. (i) This follows from Stokes' theorem. Since $\mathbb{R}^2$ is contractible, there exists a homotopy F from γ to the constant map $c : S^1 \to \mathbb{R}^2$ where $c(p) = (0,0)$ for all $p \in S^1$. By definiton,

$$F : S^1\times[0,1]\to\mathbb{R}^2$$

such that $F|_{S^1\times\{0\}} = \gamma$ and $F|_{S^1\times\{1\}} = c$. We take a particular homotopy given by $F(\alpha,t) = (1-t)\gamma(\alpha)$. Note that, we can rewrite this as a function

$$G : D^2\to\mathbb{R}^2$$

given by

$$G(x,y) = \sqrt{x^2+y^2}\cdot\gamma\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\right)$$

where D^2 is the unit disk. Note $G|_{S^1} = \gamma$. Let $\iota : S^1 \hookrightarrow D^2$ be the inclusion.

$$\begin{aligned}\int_{S^1} \gamma^*\omega = \int_{S^1} (G \circ \iota)^*\omega &= \int_{\partial D^2} \iota^*(G^*\omega) \\ &= \int_{D^2} d(G^*\omega) = \int_{D^2} G^*(d\omega) \\ &= 0.\end{aligned}$$

(ii) We use similar tricks to reduce to computing an integral over the disk and then do a change of variable to polar coordinates.

$$\begin{aligned}\int_{S^1} \iota^*((x-y^3)dx + x^3dy) &= \int_{\partial D^2} d((x-y^3)dx + x^3dy) \\ &= \int_{\partial D^2} dx \wedge dx - 3y^2dy \wedge dx + 3x^2dx \wedge dy \\ &= \int_{\partial D^2} 3(x^2+y^2)dx \wedge dy\end{aligned}$$

Setting $x = r\cos\theta$; $y = r\sin\theta$,

$$\begin{aligned}&= 3\int_{\partial D^2} r^2(\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\ &= 3\int_{\partial D^2} r^2 \cdot r dr \wedge d\theta = 3\int_{\partial D^2} r^3 dr \wedge d\theta \\ &= 3\int_0^{2\pi}\int_0^1 r^3 dr d\theta = 3\int_0^{2\pi} \frac{1}{4} d\theta \\ &= \frac{3 \times 2\pi}{4} = \frac{3}{2}\pi.\end{aligned}$$

□

3. Recall that the *suspension* of a topological space X is defined to be

$$\Sigma X = \frac{X \times [0,1]}{(x,0) \sim (y,0) \text{ and } (x,1) \sim (y,1) \text{ for all } x, y \in X}.$$

Prove that if M is a compact oriented manifold of dimension n, with finitely generated free homology groups, then ΣM cannot be a manifold unless M has the same homologies as a sphere.

Proof. First we use the Mayer-Vietoris sequence to relate the homologies of X to the homologies of ΣX. We write $\Sigma X = U \cup V$ where $U = \Sigma X \setminus [(x,1)]$ and $V = \Sigma X \setminus [(x,0)]$. Check that U and V are homotopy equivalent to a point; where as $U \cap V$ is homotopy equivalent to X. Therefore the Mayer Vietoris sequence

$$\cdots \to H_n(U \cap V) \to H_n(U) \oplus H_0(V) \to H_n(\Sigma X) \to H_{n-1}(U \cap V) \cdots$$

reduces to

$$\begin{gathered}\cdots \to H_n(X) \to H_n(pt) \oplus H_n(pt) \to H_n(\Sigma X) \to H_{n-1}(X) \to \\ H_{n-1}(pt) \oplus H_{n-1}(pt) \to H_{n-1}(\Sigma X) \to \cdots\end{gathered}$$

that is,

$$\cdots \to 0 \to H_n(\Sigma X) \to H_{n-1}(X) \to 0 \cdots \to 0 \to$$
$$H_1(\Sigma X) \to H_0(X) \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(\Sigma X) \to 0$$

from which we read off:

$$\begin{aligned} &H_i(\Sigma X) = H_{i-1}(X); \quad i \geq 2 \\ &H_1(\Sigma X) = 0 \text{ kernel of } n \mapsto (n,n) \\ &H_0(\Sigma X) = \mathbb{Z}. \end{aligned}$$

Therefore, the homologies of ΣX go as $\mathbb{Z}, 0, H_1(X), \ldots, H_n(X)$. However for a manifold, one should have the ranks are symmetric : $b_i(X) = b_{n-i}(X)$. This cannot be also true for ΣX unless $H_i(X) = 0$ for all $1 \leq i \leq n-1$. These are the homologies of a sphere. □

4. Suppose that $u \in H_n(S^n)$ and $v \in H_m(S^m)$ are generators. Also let $p_1 : S^n \times S^m \to S^n$ be the projection to the first factor and $p_2 : S^n \times S^m \to S^m$ be the projection to the second factor. Prove that $u \times v$ generate $H_{n+m}(S^n \times S^m)$. If $\mu \in H^n(S^n)$ and $\nu \in H^m(S^m)$ are the dual generators, then $\mu \times \nu$ generate $H^{n+m}(S^n \times S^m)$. Also in $H^*(S^n \times S^m)$, $p_1^*\mu \cup p_2^*\nu$ generates $H^{n+m}(S^n \times S^m)$.

Show that any map $S^4 \to S^2 \times S^2$ must induce the zero homomophism on $H_4(\ \ )$. [Hint: Use cup products].

Proof. First part is Example VI, 4.12 in Bredon.

We know that $H_4(S^4) \cong \mathbb{Z}$. Also Künneth formula implies $H_4(S^2 \times S^2) \cong \left(\oplus_{p+q=4} H_p(S^2) \otimes H_q(S^2)\right) \oplus \left(\oplus_{p+q=4} \operatorname{Tor}_1\left(H_p(S^2),\, (H_q(S^2))\right)\right)$. Since $H_p(S^2)$ is either 0 or $\mathbb{Z}$, we have $\operatorname{Tor}_1\left(H_p(S^2),\, H_q(S^2)\right) = 0$ for all p, q. Also since $H_i(S^2) = 0$ for $i \geq 3$, it is easy to see that $H_4(S^2) \cong H_2(S^2) \otimes_{\mathbb{Z}} H_2(S^2) \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$.

Now suppose the map $\varphi : S^4 \to S^2 \times S^2$ induce a map $\varphi_* = (\times d) : \mathbb{Z} \to \mathbb{Z}$. Now because of functoriality of UCT, we have the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}^1\left(H_3(S^2 \times S^2),\, \mathbb{Z}\right) & \longrightarrow & H^4(S^2 \times S^2; \mathbb{Z}) & \longrightarrow & \hom(H_4(S^2 \times S^2), \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ext}^1\left(H_3(S^4),\, \mathbb{Z}\right) & \longrightarrow & H^4(S^4; \mathbb{Z}) & \longrightarrow & \hom(H_4(S^4), \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

Using Künneth, one concludes that both the ext groups are zero. Thus we are left with a diagram

$$\begin{array}{ccc} H^4(S^2 \times S^2; \mathbb{Z}) & \xrightarrow{\cong} & \hom(H_4(S^2 \times S^2), \mathbb{Z}) \\ \varphi^* \downarrow & & \downarrow \times d \\ H^4(S^4; \mathbb{Z}) & \xrightarrow{\cong} & \hom(H_4(S^4), \mathbb{Z}) \end{array}$$

This proves that φ^* is just multiplication by d. Now we shall compute $\varphi^*(\beta \cup \gamma)$ where β and γ are generators of $H^2(S^2 \times S^2) \cong \mathbb{Z} \otimes \mathbb{Z}$ (Again use Künneth and UCT).

We claim $\beta \cup \gamma$ is non-zero in $H^4(S^2 \times S^2)$. This is Example 4.12 in Bredon. Now

$$\varphi^*(\beta \cup \gamma) = d(\beta \cup \gamma) \neq 0 \in H^4(S^n \times S^m, \mathbb{Z}).$$

On the other hand,

$$\varphi^*(\beta \cup \gamma) = \varphi^*(\beta) \cup \varphi^(\gamma) = 0$$

since $\varphi^*(\beta), \varphi^*(\gamma) \in H^2(S^4; \mathbb{Z}) = 0$. This implies that $d = 0$. □

5. Compute the cohomology of $S^2 \times S^1$ using Künneth formula and Universal Coefficient theorem. Does it match with $H^*(S^2) \otimes H^*(S^1)$?

Proof. Since the homologies are zero or $\mathbb{Z}$, the tor term of Künneth zero and hence we get

$$\begin{aligned}
H_0(S^2 \times S^1) &= \mathbb{Z} \\
H_1(S^2 \times S^1) &= H_0(S^2) \otimes H_1(S^1) \oplus H_1(S^2) \otimes H_0(S^1) = \mathbb{Z} \oplus 0 = \mathbb{Z} \\
H_2(S^2 \times S^1) &= \mathbb{Z} \otimes 0 \oplus 0 \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z} \\
H_3(S^2 \times S^1) &= \mathbb{Z} \otimes 0 \oplus 0 \otimes 0 \oplus \mathbb{Z} \otimes \mathbb{Z} \oplus 0 \otimes 0 = \mathbb{Z} \\
H_i(S^2 \times S^1) &= 0 \forall i \geq 4.
\end{aligned}$$

Here the direct sum for H_n are listed as $(0, n), (1, n-1), \ldots, (n, 0)$.

Again since $H_i(S^2)$ are projective, the ext term of UCT is zero, and hence $H^n(S^2 \times S^1; \mathbb{Z}) \cong \hom(H_n(S^2 \times S^1; \mathbb{Z})) \cong H_n(S^2 \times S^1)$, since all the $H_n(S^2 \times S^1)$ are free. Therefore,

$$H^*(S^2 \times S^1; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus 0 \oplus \cdots.$$

Now

$$\begin{aligned}
H^*(S^2) \otimes H^*(S^1) &= (\mathbb{Z} \oplus 0 \oplus \mathbb{Z} \oplus 0 \oplus \cdots) \otimes (\mathbb{Z} \oplus \mathbb{Z} \oplus 0 \oplus \cdots) \\
&= \mathbb{Z} \otimes \mathbb{Z} \oplus (\mathbb{Z} \otimes \mathbb{Z} \oplus 0 \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes 0 \oplus 0 \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes \mathbb{Z}) \oplus \\
&\qquad (\mathbb{Z} \otimes 0 \oplus 0 \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes \mathbb{Z} \oplus 0 \otimes \mathbb{Z}) \oplus 0 \cdots \\
&= \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus 0 \oplus 0 \oplus \cdots
\end{aligned}$$

Thus, $H^*(S^2 \times S^1) \cong H^*(S^2) \otimes H^*(S^1)$. □