Topology - II (End Sem)

25 April 2013

Answer all of the following questions. Try to be as precise as possible. Each question carry 20 points.

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1. Throughout this problem X is a compact, orientable manifold such that all the homology groups $H_i(X;\mathbb{Z})$ are finitely generated abelian groups.

First a few definitions. The *rank of* $H_i(X,\mathbb{Z})$ is defined to be the rank of the free part of $H_i(X;\mathbb{Z})$. For example, if $H_i(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then rank of $H_i(X;\mathbb{Z})$ is 1.

The *i-th Betti number* b_i of X is defined to be the rank of $H_i(X;\mathbb{Z})$.

Now define the *Poincaré polynomial* to be $P_X(t) = \sum_{i=0}^{\infty} b_i(X)t^i$.

Further assume that *the homology groups are finitely generated and free* for all the topological spaces in this problem.

(i) Prove that $b_i(X) = \dim_K H_i(X;K) = \dim_K H^i(X;K)$ where K is a field (of characteristic 0) and the $\dim_K$ is the dimension as a vector space. (*7 points.*)

(ii) Prove that $P_{X_1 \times X_2}(t) = P_{X_1}(t)P_{X_2}(t)$. (*6 points.*)

(iii) Prove that if $\deg P_X = n$ then $P_X(t) = t^n P_X(1/t)$. (*7 points.*)

2. (i) Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a map and ω is a *closed* 1-form on $\mathbb{R}^2$. Prove that

$$\int_{S^1} \gamma^* \omega = 0.$$

(*12 points.*)

(ii) Consider the inclusion $\iota : S^1 \hookrightarrow \mathbb{R}^2$. Let $\omega = (x - y^3)dx + x^3 dy$. Compute

$$\int_{S^1} \iota^* \omega.$$

(*8 points.*)

3. Recall that the *suspension* of a topological space X is defined to be

$$\Sigma X = \frac{X \times [0,1]}{(x,0) \sim (y,0) \text{ and } (x,1) \sim (y,1) \text{ for all } x, y \in X}.$$

Prove that if M is a compact oriented manifold of dimension n, with finitely generated, free cohomology groups, then ΣM cannot be a manifold unless M has the same homologies as a sphere. *(20 points.)*

4. Suppose that $u \in H_n(S^n)$ and $v \in H_m(S^m)$ are generators. Also let $p_1 : S^n \times S^m \to S^n$ be the projection to the first factor and $p_2 : S^n \times S^m \to S^m$ be the projection to the second factor. Prove that $u \times v$ generate $H_{n+m}(S^n \times S^m)$. If $\mu \in H^n(S^n)$ and $\nu \in H^m(S^m)$ are the dual generators, then $\mu \times \nu$ generate $H^{n+m}(S^n \times S^m)$. Also in $H^*(S^n \times S^m)$, $p_1^*\mu \cup p_2^*\nu$ generates $H^{n+m}(S^n \times S^m)$. *(6 points.)*

Show that any map $S^4 \to S^2 \times S^2$ must induce the zero homomophism on $H_4(\ \ )$. [Hint: Use cup products]. *(14 points.)*

5. Compute the cohomology of $S^2 \times S^1$ using Künneth formula and Universal Coefficient theorem. Does it match with $H^*(S^2) \otimes H^*(S^1)$? *(16 + 4 = 20 points.)*

Some useful stuff

These are just to help you remember.

Mayer-Vietoris

$$0 \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(U \cup V) \to H_{n-1}(U \cap V) \to \cdots \to H_0(U \cup V) \to 0.$$

$$\cdots \to H^n(U \cup V; G) \to H^n(U; G) \oplus H^n(V; G) \to H^n(U \cap V; G) \to H^{n+1}(U \cup V; G) \to \cdots .$$

Universal Coefficient theorem :

$$0 \to \operatorname{Ext}^1(H_{n-1}(X), G) \to H^n(X; G) \to \hom(H_n(X), G) \to 0$$

$$0 \to H_n(X) \otimes G \to H_n(X; G) \to \operatorname{Tor}_1(H_{n-1}(X), G) \to 0.$$

Künneth :

$$0 \to (H_*(X) \otimes H_*(Y))_n \to H_n(X \times Y) \to \operatorname{Tor}_1(H_*(X), H_*(Y))_{n-1} \to 0.$$

Some facts about cup products.

(i) If $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections, then $(p_X^*\alpha) \cup (p_Y^*\beta) = \alpha \times \beta$.

(ii) $\alpha \cup \beta = d^*(\alpha \times \beta)$ where $d : X \to X \times X$ is the diagonal map.