Class Test I

September 6, 2012

Answer all the questions. Total number of points: 10.

Exercise 1. Consider the map

$$F : \mathbb{R}^2 \to \mathbb{R}^2$$

given by $F(t, s) = (t + s,\ ts)$.

(1) Is this map differentiable? (*1 point.*)
(2) Find the image of this map. (*1 point.*)
(3) Find all points where DF is nonsingular. Suppose $DF(p)$ is nonsingular. Write down an function G defined in an open set containing $F(p)$ such that $G \circ F$ is identity around some open set containing p. (*2 points.*)
(4) Show that there is a maximal open set $W \subset \mathbb{R}^2$ such that F is $2 \mapsto 1$ on W, that is, each point $F(p)$, $p \in W$ is the image of exactly two distinct points in W. What happens at the other points? (*1 point.*)

Exercise 2. Consider the following maps:

(1) $C : \mathbb{R} \to \mathbb{R}^2$ given by $C(x) = (\cos x, \sin x)$,
(2) for $t \in \mathbb{R}$, $G_t : \mathbb{R} \to \mathbb{R}^2$ given by $G_t(x) = (x, t)$,
(3) for $s \in \mathbb{R}$, $H_s : \mathbb{R} \to \mathbb{R}^2$ given by $H_s(x) = (s, x)$.

Describe the images of the following three maps : $F \circ C$, $F \circ G_t$ and $F \circ H_s$, where F is the map from Exercise 1. (*1 + 1/2 + 1/2 = 2 points.*)

Exercise 3. Prove that $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = (xe^y + y, xe^y - y)$$

is a C^∞-diffeomorphism. (*3 points.*)