SOLUTIONS TO MID-SEMESTRAL EXAMINATION PROBLEMS

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(1) Prove that the subset M of the Euclidean space $\mathbb{R}^3$ which consists of all points $(x, y, z) \in \mathbb{R}^3$ satisfying

$$x^2 - y^2 + 2xz - 2yz = 1 \text{ and}$$
$$2x - y + z = 0$$

admits a structure of a C^∞ manifold. (*2 points.*)

Solution. Consider $F : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$F(x, y, z) = (x^2 - y^2 + 2xy - 2yz, 2x - y + z)$$

Then $M = F^{-1}(1, 0)$. We know that M will be a manifold if DF has full rank at every point of $F^{-1}(1, 0)$. Now

$$DF_{(a,b,c)} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix}_{(a,b,c)}$$
$$= \begin{pmatrix} 2a + 2b & -2b - 2c & 2a - 2b \\ 2 & -1 & 1 \end{pmatrix}$$

Now suppose DF has rank less than or equal to one at some point in $F^{-1}(1, 0)$. This means it satisfies the following set of equations for some k:

$$2x + 2z = 2k$$
$$2y + 2z = k$$
$$2x - 2y = k.$$

From this we get that $(x+z)-(2y+2z) = k-k = 0$ and hence $x-2y-z = 0$. Since $(x, y, z) \in F^{-1}(1, 0)$, $2x - y + z = 0$. Therefore, $3(x - y) = (x - 2y - z) + (2x - y + z) = 0$, which means $x - y = 0$. This will imply that $1 = x^2 - y^2 + 2xz - 2yz = (x - y)(x + y + 2z) = 0$ which is a contradiction. Therefore DF has rank 2 at every point of $F^{-1}(0)$ and hence M is a regular submanifold of $\mathbb{R}^3$. $\square$

(2) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x, y, z) = x^2 + y^2 - 1.$$

(a) Prove that $C = f^{-1}(0)$ is an *embedded* 2-submanifold of $\mathbb{R}^3$. (*1 point.*)

Solution. For $(x, y, z) \in f^{-1}(0)$, $x^2 + y^2 = 1$ and hence both x and y cannot be simultaneously 0. Therefore

$$Df_{(x,y,z)} = \begin{pmatrix} 2x & 2y & 0 \end{pmatrix}$$

always has rank 1 and hence C is a regular submanifold of $\mathbb{R}^3$. □

(b) Prove that a vector

$$\mathbf{v} = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right)_{(0,1,1)}$$

is tangent to C if and only if $b = 0$. (*2 points.*)

Solution. Consider the coordinate chart (U, φ) where

$$U = \{(a,b,c) \mid b > 0,\ (a,b,c) \in C\}$$
$$\varphi(a,b,c) = (a,c)$$

Note that $\varphi(U) = (-1,1) \times \mathbb{R} \subset \mathbb{R}^2$ and $\varphi^{-1}(a,c) = (a, +\sqrt{1-a^2}, c)$. Now if the coordinates of $\mathbb{R}^2$ are denoted in terms of s and t, $T_{\varphi(p)}\varphi(U)$ is generated by $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ and $T_pM = T_pU$ is generated by $(\varphi^{-1})_*\frac{\partial}{\partial s}$ and $(\varphi^{-1})_*\frac{\partial}{\partial t}$. Now for $f \in C^\infty((\varphi^{-1})_*(p))$,

$$\begin{aligned}
(\varphi^{-1})_*\frac{\partial}{\partial s}\,|_{(0,1)}\, f &= \frac{\partial}{\partial s}\,|_{(0,1)}\,(f \circ \varphi^{-1}) \\
&= \frac{\partial}{\partial s}\,|_{(0,1)}\, f(s, \sqrt{1-s^2}, t) \\
&= \frac{\partial}{\partial x}\,|_{(0,1,1)}\, f + \left.\frac{-2s}{2\sqrt{1-s^2}}\right|_{s=0,t=1} \frac{\partial}{\partial y}\,|_{(0,1,1)}\, f + 0\frac{\partial}{\partial z}\,|_{(0,1,1)}\, f \\
&= \frac{\partial}{\partial x}\,|_{(0,1,1)}\, f.
\end{aligned}$$

Similarly, one sees that $(\varphi^{-1})_*\frac{\partial}{\partial t} = \frac{\partial}{\partial z}\,|_{(0,1,1)}$. Therefore, $T_{(0,1,1)}C$ is generated by $\frac{\partial}{\partial x}\,|_{(0,1,1)}$ and $\frac{\partial}{\partial z}\,|_{(0,1,1)}$. Hence $b = 0$. □

(c) If $j : S^1 \hookrightarrow \mathbb{R}^2$ is the inclusion map of $S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$ in $\mathbb{R}^2$, prove that

$$j \times \mathrm{id}_{\mathbb{R}} : S^1 \times \mathbb{R} \to \mathbb{R}^3$$

induces a diffeomorphism from $S^1 \times \mathbb{R} \to C$. (Here, the map $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is the identity map.) (*2 points.*)

Solution. Note that $(j \times \mathrm{id}_{\mathbb{R}})((x,y),z) = (x,y,z)$. Since

$$f(j \times \mathrm{id}_{\mathbb{R}})((x,y),z) = x^2 + y^2 - 1 = 0$$

((x,y) being in S^1), we have that $j \times \mathrm{id}_{\mathbb{R}} : S^1 \times \mathbb{R} \to C \subset \mathbb{R}^3$. Note that $\psi : C \to S^1 \times \mathbb{R}$ defined by

$$\psi(x,y,z) = ((x,y),z)$$

is well defined and is a set theoretic inverse for $j \times \mathrm{id}_{\mathbb{R}}$. Now $S^1 \times \mathbb{R}$ and C are both regular submanifolds of $R^3 = \mathbb{R}^2 \times \mathbb{R}$ and the map between them is the restriction of the identity map. Thus the preferred neighbourhoods around any point of C proves that both $j \times \mathrm{id}_{\mathbb{R}}$ and ψ are differentiable. □

(3) Assume that a manifold M of dimension n admits a basis $\{X_1, \ldots, X_n\}$ for the C^∞ module $\mathfrak{X}(M)$ of C^∞ vector fields on M. (This means that at each point $p \in M$ $\{X_{1p}, \ldots, X_{np}\}$ generate the tangent space T_pM.) Prove that the function

$$F : M \times \mathbb{R}^n \to TM$$

given by $F(p) = \sum_{i=1}^n a_i X_{ip}$ is a diffeomorphism. (*4 points.*)

Solution. Consider $\Phi : M \times \mathbb{R}^n \to T_pM$ given by

$$\Phi(p, a_1, \ldots, a_n) = \sum_i a_i X_i.$$

Since $(X_i)_p$ forms a basis for T_pM, Φ is a bijection. To see that Φ is C^∞, consider a coordinate neighbourhood (U, φ) of p. Let $\pi : TM \to M$ is the map which takes a tangent vector at p to p. Consider the coordinate neighbourhood $(\pi^{-1}(U), \psi)$ in T_pM where for $V = \sum_i \alpha_i \left.\frac{\partial}{\partial x_i}\right|_p$,

$$\psi(V) = (\varphi(p), \alpha_1, \ldots, \alpha_n) \in \varphi(U) \times \mathbb{R}^n.$$

Since X_i are C^∞ vector fields there exists C^∞ functions f_{ij} on U such that $X_i = \sum_j f_{ij} \frac{\partial}{\partial x_j}$. Now

$$\begin{aligned}\psi \circ \Phi \circ (\varphi \times \mathrm{id}_{\mathbb{R}^n})^{-1}&(t_1, \ldots, t_n; a_1, \ldots, a_n)\\ &= \psi(a_i (X_i)_{\varphi(\mathbf{t})})\\ &= (t_1, \ldots, t_n; \sum_i a_i f_{i1}(\varphi^{-1}(t)), \ldots, \sum_i a_i f_{in}(\varphi^{-1}(t))).\end{aligned}$$

which is C^∞. Consider the matrix $((f_{ij}))$. Since at each point it is the matrix corresponding to change of basis, it is invertible. Let the (i,j)-th entry of the inverse be g_{ij}. g_{ij} is then C^∞ for all i and j. Then one can check that

$$\begin{aligned}\big((\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ \Phi^{-1} \circ \psi^{-1}\big)&(t_1, \ldots, t_n; a_1, \ldots, a_n)\\ &= (t_1, \ldots, t_n, \sum_i a_i g_{i1}(\varphi^{-1}(t)), \ldots, \sum_i a_i g_{in}(\varphi^{-1}(t))).\end{aligned}$$

Therefore, Φ is a diffeomorphism. □

(4) Let $S^3 \subset \mathbb{R}^4$ be the manifold

$$S^3 = \{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

Take it for granted that it is an embedded manifold in $\mathbb{R}^4$. Let $\mathbf{p} = (p_1, p_2, p_3, p_4) \in S^3$.

(a) Consider the collection (U_i^+, φ_i^+), (U_i^-, φ_i^-), where

$$U_i^+ = \{\mathbf{p} \in S^3 | p_i > 0\}$$
$$U_i^- = \{\mathbf{p} \in S^3 | p_i < 0\}$$

and $\varphi_i^\pm(\mathbf{p})$ is the vector $\mathbf{p}$ with the i-th entry removed. For example $\varphi_1^\pm(p_1, p_2, p_3, p_4) = (p_2, p_3, p_4)$. Prove that this collection is C^∞ compatible and hence defines a manifold structure on S^3. Let

$$(U, \varphi) = (U_1^+, \varphi_1^+).$$

Compute

$$(\varphi^{-1})_* \left(\sum_{i=2}^{4} a_i(x_2, x_3, x_4) \frac{\partial}{\partial x_i} \right).$$

(*2 points.*)

Solution. We shall only show that (U_1^+, φ_1^+) and (U_2^+, φ_2^+) are C^∞-compatible. All the other pairs can be proved to be C^∞-compatible in a similar way.
Note that $U_1^+ \cap U_2^+ = \{(p_1, p_2, p_3, p_4) \mid p_1 > 0, p_2 > 0\}$, $\varphi_1^+(U_1^+ \cap U_2^+) = \{(x, y, z) \mid x > 0\}$ and $\varphi_2^+(U_1^+ \cap U_2^+) = \{(s, t, u) \mid s > 0\}$. Also

$$\varphi_2^+ \circ (\varphi_1^+)^{-1}(x, y, z) = (\sqrt{1 - x^2 - y^2 - z^2}, y, z)$$

which is C^∞. Similarly,

$$\varphi_1^+ \circ (\varphi_2^+)^{-1}(s, t, u) = (\sqrt{1 - s^2 - t^2 - u^2}, t, u)$$

which is also C^∞. This completes the proof.
Now for $f \in C^\infty(\varphi^{-1}(x_2, x_3, x_4))$,

$$\begin{aligned}
&\left[(\varphi^{-1})_* \left(\sum_{i=2}^{4} a_i(\mathbf{x}) \frac{\partial}{\partial x_i} \right) \right]_{\varphi^{-1}(\mathbf{x})} f \\
&= \sum_{i=2}^{4} a_i(\mathbf{x}) \left. \frac{\partial}{\partial x_i} \right|_{\varphi^{-1}(\mathbf{x})} (f \circ \varphi^{-1}) \\
&= \sum_{i=2}^{4} a_i(\mathbf{x}) \left. \frac{\partial}{\partial x_i} \right|_{\varphi^{-1}(\mathbf{x})} f \left(\sqrt{1 - x_2^2 - x_3^2 - x_4^2}, \mathbf{x} \right) \\
&= \sum_{i=2}^{4} a_i(\mathbf{x}) \left[\frac{-2x_i}{2\sqrt{1 - x_2^2 - x_3^2 - x_4^2}} \left. \frac{\partial}{\partial x_1} \right|_{\varphi^{-1}(\mathbf{x})} f + \left. \frac{\partial}{\partial x_i} \right|_{\varphi^{-1}(\mathbf{x})} f \right]
\end{aligned}$$

where $\mathbf{x} = (x_2, x_3, x_4)$. Rearranging we get,

$$\begin{aligned}
(\varphi^{-1})_* \left(\sum_{i=2}^{4} a_i(x_2, x_3, x_4) \frac{\partial}{\partial x_i} \right) = \\
- \sum_{i=2}^{4} \frac{x_i a_i(x_2, x_3, x_4)}{\sqrt{1 - x^2 - x^3 - x^4}} \frac{\partial}{\partial x_1} + a_2(x_2, x_3, x_4) \frac{\partial}{\partial x_2} \\
+ a_3(x_2, x_3, x_4) \frac{\partial}{\partial x_3} + a_4(x_2, x_3, x_4) \frac{\partial}{\partial x_4}.
\end{aligned}$$

□

(b) Suppose $b_i(x_2, x_3, x_4)$ be the coefficients of $\frac{\partial}{\partial x_i}$ in the above expression, that is

$$\sum_{i=1}^{4} b_i(x_2, x_3, x_4) \frac{\partial}{\partial x_i} = (\varphi^{-1})_* \left(\sum_{i=2}^{4} a_i(x_2, x_3, x_4) \frac{\partial}{\partial x_i} \right).$$

Prove that for $\mathbf{q} \in U$, $\sum_{i=1}^{4} q_i b_i(q_2, q_3, q_4) = 0$. (*1 point.*)

Solution.

$$\sum_{i=1}^{4} q_i b_i(q_2, q_3, q_4)$$
$$= -q_1 \left(\sum_{i=2}^{4} \frac{q_i a_i(\mathbf{q})}{\sqrt{1 - q_2^2 - q_3^2 - q_4^2}} \right) + q_2 a_2(\mathbf{q}) + q_3 a_3(\mathbf{q}) + q_4 a_4(\mathbf{q})$$
$$= 0$$

since $q_1 = \sqrt{1 - q_2^2 - q_3^2 - q_4^2}$. □

(c) Conversely if b_1, b_2, b_3 and b_4 are C^∞ functions from $\varphi(U)$ to $\mathbb{R}$, such that

$$\sum_{i=1}^{4} q_i b_i(q_2, q_3, q_4) = 0 \text{ for all } \mathbf{q} \in U$$

then prove that

$$\sum_{i=1}^{4} b_i(q_2, q_3, q_4) \left(\frac{\partial}{\partial x_i} \right)_{\mathbf{q}}$$

is a tangent vector at every $\mathbf{q} \in U \subset S^3$. (*2 points.*)

Solution. If $\sum_{i=1}^{4} q_i b_i(q_2, q_3, q_4) = 0$, then

$$b_1(q_2, q_3, q_4) = -\sum_{i=2}^{4} \frac{q_2 b_2(\mathbf{q})}{q_1} = -\sum_{i=2}^{4} \frac{q_2 b_2(\mathbf{q})}{\sqrt{1 - q_2^2 - q_3^2 - q_4^2}}$$

Thus by the computation done in 4a, we see that

$$\sum_{i=1}^{4} b_i(\mathbf{q}) \frac{\partial}{\partial x_i} = (\varphi^{-1})_* \sum_{i=2}^{4} b_i(\mathbf{q}) \frac{\partial}{\partial x_i}$$

and hence is a tangent to S^3 at (q_1, q_2, q_3, q_4). □

(d) Assuming the previous problem, prove that the vector fields given by

$$X_p = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} \right)_{\mathbf{p}}$$
$$Y_p = \left(-z\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - y\frac{\partial}{\partial t} \right)_{\mathbf{p}}$$
$$Z_p = \left(-t\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t} \right)_{\mathbf{p}}$$

for $\mathbf{p} \in S^3$ define a diffeomorphism $S^3 \times \mathbb{R}^3 \to TS^3$. (*1 point.*)

Solution. Using 4c, we see that for each of the vectors given, $\sum_i p_i b_i = 0$ and hence they are tangent to S^3 at $\mathbf{p}$.
Consider the map $\Phi : S^3 \times \mathbb{R}^3 \to TS^3$ given by

$$\Phi(\mathbf{p}, a_1, a_2, a_3) = a_1 X_{\mathbf{p}} + a_2 Y_{\mathbf{p}} + a_3 Z_{\mathbf{p}}.$$

It is easy to check that $X_{\mathbf{p}}$, $Y_{\mathbf{p}}$ and $Z_{\mathbf{p}}$ are linearly independent and hence one can define an inverse $\Psi : TS^3 \to S^3 \times \mathbb{R}^3$ by

$$\Psi(W_{\mathbf{p}}) = (\mathbf{p}, w_1, w_2, w_3)$$

where $W_{\mathbf{p}} = w_1 X_{\mathbf{p}} + w_2 Y_{\mathbf{p}} + w_3 Z_{\mathbf{p}}$. Using the fact that X, Y and Z are C^∞ vector fields on S^3 and projection maps are C^∞, we see that both Φ and Ψ and C^∞. □

(5) Consider the vector fields

$$X = xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial z}; \qquad Y = y\frac{\partial}{\partial y}$$

on $\mathbb{R}^3$ and the map $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 y$. Compute

(a) $(fX)_{(1,1,0)}$ (*1 point.*),

Solution.

$$\begin{aligned}(fX)_{(1,1,0)} &= f(1,1,0)X_{(1,1,0)} = X_{(1,1,0)} \\ &= 1.1\left.\frac{\partial}{\partial x}\right|_{(1,1,0)} + 1^2\left.\frac{\partial}{\partial z}\right|_{(1,1,0)} \\ &= \left.\frac{\partial}{\partial x}\right|_{(1,1,0)} + \left.\frac{\partial}{\partial z}\right|_{(1,1,0)}.\end{aligned}$$

□

(b) $(Xf)(1, 1, 0)$ (*1 point.*),

Solution.

$$\begin{aligned}(Xf)(1,1,0) &= \left(xy\frac{\partial}{\partial x}(x^2y) + x^2\frac{\partial}{\partial z}(x^2y)\right)(1,1,0) \\ &= (xy.2xy + 0)(1,1,0) = 2\end{aligned}$$

□

(c) $f_*(X_{(1,1,0)})$ (*1 point.*).

Solution. Note that $f_*(X|_{(1,1,0)})$ is a tangent vector at $f(1,1,0) = 1 \in \mathbb{R}$. Suppose $T_1\mathbb{R}$ is generated by $\left.\frac{d}{dt}\right|_{t=1}$. Then for $g \in C^\infty(1)$, we have

$$\begin{aligned}f_*(X|_{(1,1,0)})g &= X|_{(1,1,0)}(g \circ f) \\ &= \left(1\left.\frac{\partial}{\partial x}\right|_{(1,1,0)} + 1\left.\frac{\partial}{\partial z}\right|_{(1,1,0)}\right) g(x^2y) \\ &= \left.\frac{\partial}{\partial x}\right|_{(1,1,0)} g(x^2y) + 0 \\ &= \left.\frac{d}{dt}\right|_{t=1}(g)\left.\frac{\partial}{\partial x}\right|_{(1,1,0)}(x^2y) = (2xy)(1,1,0)\left.\frac{d}{dt}\right|_{t=1}(g) \\ &= 2\left.\frac{d}{dt}\right|_{t=1}(g).\end{aligned}$$

From this we conclude that

$$f_*(X_{(1,1,0)}) = 2\left.\frac{d}{dt}\right|_{t=1}.$$

□