MID-SEMESTRAL EXAMINATION: IISER PUNE

Differential Geometry: September 27, 2012, 11:00 AM - 12:30 PM

Answer all the 5 questions.
Maximum score: 20

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(1) Prove that the subset M of the Euclidean space $\mathbb{R}^3$ which consists of all points $(x, y, z) \in \mathbb{R}^3$ satisfying
$$x^2 - y^2 + 2xz - 2yz = 1 \text{ and}$$
$$2x - y + z = 0$$
admits a structure of a C^∞ manifold. (*2 points.*)

(2) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by
$$f(x, y, z) = x^2 + y^2 - 1.$$
(a) Prove that $C = f^{-1}(0)$ is an *embedded* 2-submanifold of $\mathbb{R}^3$. (*1 point.*)
(b) Prove that a vector
$$\mathbf{v} = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \right)_{(0,1,1)}$$
is tangent to C if and only if $b = 0$. (*2 points.*)
(c) If $j : S^1 \hookrightarrow \mathbb{R}^2$ is the inclusion map of $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ in $\mathbb{R}^2$, prove that
$$j \times \mathrm{id}_{\mathbb{R}} : S^1 \times \mathbb{R} \to \mathbb{R}^3$$
induces a diffeomorphism from $S^1 \times \mathbb{R} \to C$. (Here, the map $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is the identity map.) (*2 points.*)

(3) Assume that a manifold M of dimension n admits a basis $\{X_1, \ldots, X_n\}$ for the C^∞ module $\mathfrak{X}(M)$ of C^∞ vector fields on M. (This means that at each point $p \in M$ $\{X_{1p}, \ldots, X_{np}\}$ generate the tangent space T_pM.) Prove that the function
$$F : M \times \mathbb{R}^n \to TM$$
given by $F(p) = \sum_{i=1}^n a_i X_{ip}$ is a diffeomorphism. (*4 points.*)

(4) Let $S^3 \subset \mathbb{R}^4$ be the manifold
$$S^3 = \{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$
Take it for granted that it is an embedded manifold in $\mathbb{R}^4$. Let $\mathbf{p} = (p_1, p_2, p_3, p_4) \in S^3$.
(a) Consider the collection (U_i^+, φ_i^+), (U_i^-, φ_i^-), where
$$U_i^+ = \{\mathbf{p} \in S^3 | p_i > 0\}$$
$$U_i^- = \{\mathbf{p} \in S^3 | p_i < 0\}$$

and $\varphi_i^{\pm}(\mathbf{p})$ is the vector $\mathbf{p}$ with the i-th entry removed. For example $\varphi_1^{\pm}(p_1, p_2, p_3, p_4) = (p_2, p_3, p_4)$. Prove that this collection is C^∞ compatible and hence defines a manifold structure on S^3. Let

$$(U, \varphi) = (U_1^+, \varphi_1^+).$$

Compute

$$(\varphi^{-1})_* \left(\sum_{i=2}^{4} a_i(x_2, x_3, x_4) \frac{\partial}{\partial x_i} \right).$$

(*2 points.*)

(b) Suppose $b_i(x_2, x_3, x_4)$ be the coefficients of $\frac{\partial}{\partial x_i}$ in the above expression, that is

$$\sum_{i=1}^{4} b_i(x_2, x_3, x_4) \frac{\partial}{\partial x_i} = (\varphi^{-1})_* \left(\sum_{i=2}^{4} a_i(x_2, x_3, x_4) \frac{\partial}{\partial x_i} \right).$$

Prove that for $\mathbf{q} \in U$, $\sum_{i=1}^{4} q_i b_i(q_2, q_3, q_4) = 0$. (*1 point.*)

(c) Conversely if b_1, b_2, b_3 and b_4 are C^∞ functions from $\varphi(U)$ to $\mathbb{R}$, such that

$$\sum_{i=1}^{4} q_i b_i(q_2, q_3, q_4) = 0 \text{ for all } \mathbf{q} \in U$$

then prove that

$$\sum_{i=1}^{4} b_i(q_2, q_3, q_4) \left(\frac{\partial}{\partial x_i} \right)_{\mathbf{q}}$$

is a tangent vector at every $\mathbf{q} \in U \subset S^3$. (*2 points.*)

(d) Assuming the previous problem, prove that the vector fields given by

$$X_p = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} \right)_{\mathbf{p}}$$

$$Y_p = \left(-z\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - y\frac{\partial}{\partial t} \right)_{\mathbf{p}}$$

$$Z_p = \left(-t\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t} \right)_{\mathbf{p}}$$

for $\mathbf{p} \in S^3$ define a diffeomorphism $S^3 \times \mathbb{R}^3 \to TS^3$. (*1 point.*)

(5) Consider the vector fields

$$X = xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial z}; \qquad Y = y\frac{\partial}{\partial y}$$

on $\mathbb{R}^3$ and the map $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 y$. Compute

(a) $(fX)_{(1,1,0)}$ (*1 point.*),
(b) $(Xf)(1, 1, 0)$ (*1 point.*),
(c) $f_*(X_{(1,1,0)})$ (*1 point.*).