ASSIGNMENT 1

This assignment is due on August 29, 2012.

1. Implicit function thoerem

Definition 1. Suppose $F : U \to \mathbb{R}^m$, U being an open subset of $\mathbb{R}^n$, be C^r, $r \geq 1$ or $r = \infty$. Define the *rank of F at $\vec{x}$* to be the rank of the matrix $DF(\vec{x})$.

In the following set of exercises we shall prove the following theorem.

Theorem 1. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be open sets. Let $F : A \to B$ be a C^r mapping. Suppose that* $\operatorname{rank} F$ *at $\vec{x}$ is k for all $\vec{x} \in A$. If $\vec{a} \in A$ and $\vec{b} = F(\vec{a}) \in B$, then there exists open sets $A_0 \subset A$ and $B_0 \subset B$ such that $\vec{a} \in A_0$ and $\vec{b} \in B_0$; and there exists C^r diffeomorphisms*

$$G : A_0 \to U \subset \mathbb{R}^n \text{ (open,)}$$

and

$$H : B_0 \to V \subset \mathbb{R}^m \text{ (open,)}$$

such that $H \circ F \circ G^{-1}(U) \subset V$ and

$$H \circ F \circ G^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0).$$

Exercise 1. Prove that it is enough to prove the statement for F, $\vec{a}$ and $\vec{b}$ satisfying

(1) $\vec{a} = \vec{b} = \vec{0}$;
(2) $F(\vec{0}) = \vec{0}$; and
(3) if $F(\vec{x}) = (f_1(\vec{x}), \dots, f_n(\vec{x}))$, the top left $k \times k$ block of the Jacobian,

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_k} \end{pmatrix}$$

is non-singular.

From now on assume that F satisfies the properties listed in exercise 1.

Exercise 2. Define $G : A \to \mathbb{R}^n$ by

$$G(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_k(x_1, \dots, x_n), x_{k+1}, \dots, x_n)\,.$$

Show that there exists an open neighbourhood A_1 of $\vec{0}$ in A such that G is a diffeomorphism onto U_1 where $U_1 := G(A_1)$.

Exercise 3. Consider $F \circ G^{-1} : U_1 \to B$.

(1) Compute $D(F \circ G^{-1})(\vec{x})$, $x \in U_1$.
(2) Prove that $\operatorname{rank} D(F \circ G^{-1})(\vec{x})$ is k on U_1.
(3) If $F \circ G^{-1}(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = (y_1, \dots, y_k, y_{k+1}, \dots, y_m)$, show that
   (a) $y_i = x_i$ for $1 \leq i \leq k$, and

(b) if for $k+1 \leq i \leq m$, $\bar{f}_i(\vec{x})$ denotes y_i, show that $\bar{f}_i$ depends only on $x_1, \ldots, x_k$.

Exercise 4. Define $T : V_1 \to \mathbb{R}^m$, for some $V_1 \subset \mathbb{R}^m$ with $\vec{0} \in V_1$ by

$$T(y_1, \ldots, y_k, y_{k+1}, \ldots, y_m) = \big(y_1, \ldots, y_k, y'_{k+1}, \ldots, y'_m\big)$$

where

$$y'_j = y_j + \bar{f}_j(y_1, \ldots, y_k), \text{ for } k+1 \leq j \leq m$$

Show that there exists a neighbourhood V_1 of $\vec{0}$ such that

(1) for all $\vec{y} \in V_1$, $\bar{f}_j(y_1, \ldots, y_k)$ are defined for $k+1 \leq j \leq m$, and
(2) $T(V_1) \subset B$.

Exercise 5. Check that $T(\vec{0}) = \vec{0}$ and T is nonsingular at all points of V_1.

Exercise 6. Put the results from the above results together to construct a proof of theorem 1.

Exercise 7. Assume implicit function theorem (theorem 1) and prove inverse function theorem (theorem 6.4, page 42 in Boothby).

2. Some other problems

Exercise 8. Prove that the following are topological manifolds

(1) S^1,
(2) S^2,
(3) $\mathbb{RP}^2$.

Exercise 9. Suppose K is a convex, compact subset of $\mathbb{R}^n$. Suppose $T : K \to K$ be such that $\|T(x_1) - T(x_2)\| \leq \lambda \|x_1 - x_2\|$ for all $x_1, x_2 \in K$ and $0 \leq \lambda \leq 1$. (Note that it differs from contraction mapping, since we are allowing $\lambda = 1$.) Prove that T has a fixed point. Give an example showing that the convexity is necessary.

Exercise 10. Give an example (with proof) of a locally Euclidean space which is not Hausdorff.