Solutions: MTH413: Differential Topology
Indian Institute of Science Education and Research, Pune

End Sem Exam

November 24, 2012, 9 AM-12 noon

Hints for the solutions.

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1. In each of the following examples, exactly one of the defining conditions of a topological manifold fails for M. State that condition and explain why it fails.

(i) $M = X/\sim$ where $X = \mathbb{R}\times\{0\}\cup\mathbb{R}\times\{1\}\subset\mathbb{R}^2$. $\sim$ is the equivalence relation on X generated by $(x,0)\sim(x,1)$ for all $x\neq 0$. (*1 point.*)

(ii) $M =$ Disjoint union of uncountable copies of $\mathbb{R}$. (*1 point.*)

(iii) $M = \{(x,y)\in\mathbb{R}^2 \mid xy=0\}$. (*1 point.*)

Solution. For the first one Hausdorffness fails as one cannot separate $[(x,0)]$ and $[(x,1)]$. For the second one there are uncountably ($\mathbb{R}$ many) disjoint open sets and hence it cannot have a countable basis. For the third, M is not locally Euclidean around $(0,0)$. □

2. Consider the mapping $f:\mathbb{R}^3\to\mathbb{R}^2$ given by

$$f(x,y,z) = (x^2+y, x^2+y^2+z^2+y).$$

Show that $f^{-1}((0,1))$ is an embedded submanifold of $\mathbb{R}^3$. (*1 point.*)

Show that $f^{-1}((0,1))$ is C^1-diffeomorphic to S^1. (*2 points.*)

Solution. It is clear that $Df_{(a,b,c)}$ has rank two by considering the two cases when a is zero (in which case for any point in $f^{-1}(0,1)$, c cannot be zero) and a is nonzero in which it is evident that the two rows are linearly independent.

To construct the C^1 diffeomorphism just observe that $f^{-1}(0,1)$ is diffeomorphic to $\{(x,y,z)\in\mathbb{R}^3 \mid x^4+z^2=1, y=-x^2\}$. Then define $g(x)=\operatorname{sgn}(x)x^2$ and then look at the map $\Phi(x,y,z)=(g(x),z)$ and show that it is a C^1 diffeomorphism. □

3. Prove that the annulus

$$A = \{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 4\}$$

is a smooth manifold with boundary. What is the boundary? (*2+1=3 points.*)

Solution. Check that the interior of the annulus is

$$A^\circ = \{(x,y)\in\mathbb{R}^2 \mid 1< x^2+y^2< 4\}.$$

Then give explicit coordinate charts at the boundary at $(n\cos\theta, n\sin\theta)$ by $U = \{(rn\cos(\theta+\varphi), rn\sin(\theta+\varphi)) \mid \cdots\}$ for small values of r and φ ($n=1,2$), and map it to $(r,\varphi)\in\mathbb{R}^2$. Now the boundary is clear. □

4. Let $i: S^1\hookrightarrow\mathbb{R}^2$ be the inclusion of the unit circle in $\mathbb{R}^2$.

(i) Let $\omega = -ydx + xdy$ be a 1-form on $\mathbb{R}^2$. Compute

$$\int_{S^1} i^*\omega.$$

(*1 point.*)

(ii) Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$. Let ω be the 1-form

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Prove that there is no smooth function $f : M \to \mathbb{R}$ such that $\omega = df$.

(*2 points.*)

Solution. The first part is a straight forward computation. The answer is 2π.

For the second part, a similar computation prove that if $\varphi : S^1 \to \mathbb{R}^2$ is an inclusion, then $\int_{S^1} \varphi^*\omega = 2\pi$. On the other hand, if $\omega = df$, Stokes' theorem will imply that the integral is the same as $\int_{\partial D} \varphi^* df = (-1)^0 \int_D \varphi^*(d \circ df) = 0$, which is a contradiction. □

5. Let f be a C^∞ function on $\mathbb{R}^2$. Suppose D is a compact connected subset of $\mathbb{R}^2$ which is a domain of integration. Suppose that $f|_{\partial D} = 0$. Prove that

$$\int_D f\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy = -\int_D \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dx \wedge dy.$$

(*3 points.*)

Solution. Use Stokes' theorem on the one form $G = f\frac{\partial f}{\partial x}dy - f\frac{\partial f}{\partial y}dx$ and integrate on ∂D. This integral is zero by the assumption. On the other hand, Stokes' theorem will give RHS - LHS. □

6. Prove that the set

$$S = \left\{(x,y,z) \mid x^3 - y^3 + xyz - xy = 1\right\} \subset \mathbb{R}^3$$

is a differentiable manifold embedded in $\mathbb{R}^3$. Find the tangent space $T_{(1,1,2)}S$ of the surface at $p = (1,1,2)$. (*3 points.*)

Solution. It is easy to check that DF is of rank one where F is the defining equation for S. Now using the fact that the vectors in the tangent should be perpendicular to the gradient, one gets that the tangent space is $\left\{a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \mid 4a - 2b + c = 0\right\}$. □

7. Consider the 1-form

$$\alpha = xdy - ydx + zdt - tdz$$

on $\mathbb{R}^4$ and let $i : S^3 = \left\{(x,y,z,t) \mid x^2+y^2+z^2+t^2 = 1\right\} \hookrightarrow \mathbb{R}^4$ be the inclusion. Prove that $i^*\alpha$ does not vanish on the sphere S^3. (*3 points.*)

Solution. Just evaluate it on the nowhere vanishing vector field $(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z})$. □

8. A form ω is said to be *closed* if $d\omega = 0$. An r-form ω is said to be *exact* if there exists an $r-1$-form θ such that $\omega = d\theta$. Prove the following.

(i) If α and β are closed differential forms then so is $\alpha \wedge \beta$. (*1.5 points.*)

(ii) Moreover, in addition to the assumptions in 8(i), if β is exact then $\alpha \wedge \beta$ is exact. (*1.5 points.*)

Solution. For both the problems the generalized Liebnitz rule is enough to conclude. □

9. Prove that if $f : M \to \mathbb{R}$ is a nowhere vanishing function and if ω is a 1-form on M such that $d(f\omega) = 0$, then

$$\omega \wedge d\omega = 0.$$

(*3 points.*)

Solution. Expanding $d(f\omega)$ using Liebnitz rule, one concludes that $fd\omega = df \wedge \omega$. From this it follows that $fd\omega \wedge \omega = 0$ and now the conclusion follows from the fact that f is nowhere vanishing. □

10. For an r-form ω on M, and a vector field X on M, define

$$i_X\omega(X_2, \ldots, X_r) = \omega(X, X_1, \ldots, X_r)$$

for vector fields $X_2, \ldots, X_r$ on M. Prove that

(i) $i_X\omega$ is an $(r-1)$-form. (*1 point.*)
(ii) $i_X \circ i_X = 0$. (*0.5 point.*)
(iii) $i_X(\omega \wedge \sigma) = (i_X\omega) \wedge \sigma + (-1)^r \omega \wedge (i_X\sigma)$. (*1.5 points.*)

Solution. All of these are straight forward computations using properties of forms. □

11. Recall that a manifold M is said to be *parallelizable* if there exists globally defined vector fields $E_1, \ldots, E_n$ which generate the tangent space at each point $p \in M$. Prove that a parallelizable manifold is orientable. (*3 points.*)

Solution. Define an n form Ω by imposing the condition that $\Omega(E_1, \ldots, E_n) = 1$ at all points and then check that it is a smooth n form. □