MTH413: Differential Topology
Indian Institute of Science Education and Research, Pune

End Sem Exam
November 24, 2012, 9 AM-12 noon

Answer any 10 of the following questions.

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1. In each of the following examples, exactly one of the defining conditions of a topological manifold fails for M. State that condition and explain why it fails.

(i) $M = X/\sim$ where $X = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\} \subset \mathbb{R}^2$. $\sim$ is the equivalence relation on X generated by $(x, 0) \sim (x, 1)$ for all $x \neq 0$. (*1 point.*)

(ii) $M =$ Disjoint union of uncountable copies of $\mathbb{R}$. (*1 point.*)

(iii) $M = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$. (*1 point.*)

2. Consider the mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$f(x, y, z) = (x^2 + y, x^2 + y^2 + z^2 + y).$$

Show that $f^{-1}((0, 1))$ is an embedded submanifold of $\mathbb{R}^3$. (*1 point.*)

Show that $f^{-1}((0, 1))$ is C^1-diffeomorphic to S^1. (*2 points.*)

3. Prove that the annulus

$$A = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}$$

is a smooth manifold with boundary. What is the boundary? (*2+1=3 points.*)

4. Let $i : S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion of the unit circle in $\mathbb{R}^2$.

(i) Let $\omega = -ydx + xdy$ be a 1-form on $\mathbb{R}^2$. Compute

$$\int_{S^1} i^*\omega.$$

(*1 point.*)

(ii) Let $M = \mathbb{R}^2 \setminus \{(0, 0)\}$. Let ω be the 1-form

$$\omega = -\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy.$$

Prove that thre is no smooth function $f : M \to \mathbb{R}$ such that $\omega = df$. (*2 points.*)

5. Let f be a C^∞ function on $\mathbb{R}^2$. Suppose D is a compact connected subset of $\mathbb{R}^2$ which is a domain of integration. Suppose that $f|_{\partial D} = 0$. Prove that

$$\int_D f\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy = -\int_D \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dx \wedge dy.$$

(*3 points.*)

6. Prove that the set

$$S = \{(x, y, z) \mid x^3 - y^3 + xyz - xy = 1\} \subset \mathbb{R}^3$$

is a differentiable manifold embedded in $\mathbb{R}^3$. Find the tangent space $T_{(1,1,2)}S$ of the surface at $p = (1, 1, 2)$. (*3 points.*)

7. Consider the 1-form

$$\alpha = xdy - ydx + zdt - tdz$$

on $\mathbb{R}^4$ and let $i : S^3 = \{(x, y, z, t) \mid x^2 + y^2 + z^2 + t^2 = 1\} \hookrightarrow \mathbb{R}^4$ be the inclusion. Prove that $i^*\alpha$ does not vanish on the sphere S^3. (*3 points.*)

8. A form ω is said to be *closed* if $d\omega = 0$. An r-form ω is said to be *exact* if thre exists an $r - 1$-form θ such that $\omega = d\theta$. Prove the following.

(i) If α and β are closed differential forms then so is $\alpha \wedge \beta$. (*1.5 points.*)

(ii) Moreover, in addition to the assumptions in 8(i), if β is exact then $\alpha \wedge \beta$ is exact. (*1.5 points.*)

9. Prove that if $f : M \to \mathbb{R}$ is a nowhere vanishing function and if ω is a 1-form on M such that $d(f\omega) = 0$, then

$$\omega \wedge d\omega = 0.$$

(*3 points.*)

10. For an r-form ω on M, and a vector field X on M, define

$$i_X\omega(X_2, \ldots, X_r) = \omega(X, X_1, \ldots, X_r)$$

for vector fields $X_2, \ldots, X_r$ on M. Prove that

(i) $i_X\omega$ is an $(r - 1)$-form. (*1 point.*)

(ii) $i_X \circ i_X = 0$. (*0.5 point.*)

(iii) $i_X(\omega \wedge \sigma) = (i_X\omega) \wedge \sigma + (-1)^r \omega \wedge (i_X\sigma)$. (*1.5 points.*)

11. Recall that a manifold M is said to be *parallelizable* if there exists globally defined vector fields $E_1, \ldots, E_n$ which generate the tangent space at each point $p \in M$. Prove that a parallelizable manifold is orientable. (*3 points.*)