Assignment II

Due date: Nov 16, 2012

Total number of points: 30
Weightage: 5

Throughout this assignment, V is a vector space over $\mathbb{R}$ of dimension n.

Exercise 1. Prove that $\mathscr{T}_s^r$ is a vector space over $\mathbb{R}$ of dimension n^{r+s}.

Exercise 2. Let $\mathscr{A}$ and $\mathscr{S}$ be the maps $\mathscr{T}^r \to \mathscr{T}^r$ defined by

$$\mathscr{S}\Phi(v_1, \ldots, v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \Phi(v_{\sigma(1)}, \ldots, v_{\sigma(r)})$$

$$\mathscr{A}\Phi(v_1, \ldots, v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\Phi(v_{\sigma(1)}, \ldots, v_{\sigma(r)}).$$

Prove that

(i) $\mathscr{A} \circ \mathscr{A} = \mathscr{A}$ and $\mathscr{S} \circ \mathscr{S} = \mathscr{S}$.

(ii) $\mathscr{A}(\mathscr{T}^r(V)) = \Lambda^r(V)$ and $\mathscr{S}(\mathscr{T}^r(V)) = \Sigma^r(V)$.

(iii) Φ is alternating if and only if $\mathscr{A}(\Phi) = \Phi$. Also, Φ is symmetric if and only if $\mathscr{S}(\Phi) = \Phi$.

(iv) If $F : V \to W$ is a linear map of $\mathbb{R}$ vector spaces, and $F^* : \mathscr{T}^r(W) \to \mathscr{T}^r(V)$ be the corresponding pull back map on forms, then

$$\mathscr{A} \circ F^* = F^* \circ \mathscr{A}, \qquad \mathscr{S} \circ F^* = F^* \circ \mathscr{S}.$$

(v) The maps $\mathscr{A}$ and $\mathscr{S}$ are defined on $\mathscr{T}^r(M)$ and they satisfy all the above properties, where in (iv), $F^* : \mathscr{T}^r(N) \to \mathscr{T}^r(M)$ is induced by a C^∞-map $F : M \to N$.

Exercise 3. Let M be a manifold and (V, θ) be a coordinate neighbourhood. Let $E_i = (\theta^{-1})_* \frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$ be the corresponding coordinate frame. Let $\omega_1, \ldots, \omega_n$ be the dual coordiante coframe. The prove that for all $\varphi \in \mathscr{T}^r(V)$, there exists a *unique* expression

$$\varphi = \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} a_{i_1,\ldots,i_r} \omega_{i_1} \otimes \cdots \otimes \omega_{i_r}$$

where $a_{i_1,\ldots,i_r} : V \to \mathbb{R}$ are C^∞ functions.

Exercise 4. Recall that the *exterior product* $\Lambda^r(V) \times \Lambda^s(V) \to \Lambda^{r+s}(V)$ is given by

$$(\varphi, \psi) \mapsto \varphi \wedge \psi = \frac{(r+s)!}{r!s!} \mathscr{A}(\varphi \otimes \psi).$$

Prove that $\wedge$ is bilinear and associative.

Exercise 5. With V and n as above,

(i) Show that, if $r > n$ then $\Lambda^r(V) = 0$.

(ii) For $0 \le r \le n$, show that $\dim \Lambda^r(V) = \binom{n}{r}$ with a basis given by

$$\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_r} \mid 1 \le i_1 < \cdots < i_r \le n\}$$

where $\omega_1, \ldots, \omega_n$ form a basis for $\Lambda^1(V) = V^*$.

Exercise 6. Recall that an *exterior differential form of degree* r or an r *form* is an alternating covariant tensor field. Let $\Lambda(M) = \Lambda^0(M) \oplus \Lambda^1(M) \oplus \cdots$ be the $\mathbb{R}$ vector space of all differential forms.

(i) Show that for $\varphi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$ the formula

$$(\varphi \wedge \psi)_p = \varphi_p \wedge \psi_p$$

defines an associative product on $\Lambda(M)$ satisfying

$$\varphi \wedge \psi = (-1)^{rs} \psi \wedge \varphi.$$

which makes $\Lambda(M)$ into an $\mathbb{R}$ algebra.

(ii) If $f \in C^\infty(M)$, then $(f\varphi) \wedge \psi = \varphi \wedge (f\psi) = f(\varphi \wedge \psi)$.

(iii) If $\omega_1, \ldots, \omega_n$ is a field of coframes on an open set $U \subset M$, then

$$\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_r} \mid 1 \le i_1 < \cdots < i_r \le n\}$$

forms a basis for $\Lambda^r(M)$.

(iv) If $F : M \to N$ is C^∞, then $F^* : \Lambda(N) \to \Lambda(M)$ is a homomorphism of algebras.