RIEMANN SURFACES AND THEIR MODULI SPACES

COURSE OUTLINE

The central objects of this minicourse are

compact Riemann surfaces

which are closed complex manifolds of dimension 1 (or equivalently, non-singular complex projective curves). Vamsi's course on complex manifolds and vector bundles is ambivalent towards dimension; however in this course, we will see that dimension 1 enjoys some special features, and a motivation of a lot of work in higher dimensional complex manifolds is to generalize and find analogues of these features.
The feature that we are going to focus on in this minicourse is the fact that there are two perspectives of viewing Riemann surfaces:

$$\text{complex analysis} \longleftarrow \text{compact Riemann surfaces} \longrightarrow \text{hyperbolic geometry}$$

which are useful by themselves, and the interaction between them is also productive. In the course of the lectures we shall describe these arrows, and objects on either side – one one side, we have the canonical bundle, harmonic and holomorphic differentials, periods; on the other, we have pants decomposition, and length-and-twist parameters. Both arrows require some work: one direction is the Uniformization theorem, the other requires an existence theorem of harmonic forms, which marks the beginnings of Hodge theory.

One basic problem is

Classification problem: What are all the compact Riemann surfaces up to biholomorphism?

and as a solution of this "moduli problem" it turns out we have one discrete invariant that is the topological genus g, and for each g we have $\mathcal{M}_g$, the moduli space of Riemann surfaces.
It turns out this moduli space can be described and studied from both the complex analytic perspective and the hyperbolic geometric perspective, and we shall see how these are both useful. In particular, $\mathcal{M}_g$ has dimension $6g-6$ over $\mathbb{R}$ for $g \geq 2$, and one of the goals of the course is to see proofs of this.

Here's the outline for the lectures:

- *Lecture 1:* Definition of Riemann surface, examples, and existence of holomorphic differentials via harmonic forms and the Hodge decomposition, modulo the proof of Weyl's Lemma.
  References: §5.1,5.2 of Jost's book on "Compact Riemann surfaces".

- *Lecture 2:* Conformal metric, the hyperbolic plane: geodesics, isometries, examples of Fuchsian groups uniqueness of geodesics, pair of pants decomposition, length spectrum and finiteness of automorphisms.
  References: J. Aramayona's notes on "Hyperbolic structures on surfaces",

available on his webpage, Farb-Margalit's book "A Primer on Mapping Class Groups".

- *Lectures 3, 4:* Definition of moduli space and Teichmüller space, the notion of a marking, Teichmüller space of a torus, uniqueness of hyperbolic pair of pants with prescribed boundary lengths, Fenchel-Nielsen parametrization, the definition of a mapping class group and the proof that the action of the mapping class group on $\mathcal{T}_g$ is properly discontinuous, and the construction of $\mathcal{M}_g$ (from the hyperbolic geometry viewpoint) as a quotient.
  References: §3.5, 3.6 of Hubbard's "Teichmüller theory" and Chapter 12 of Farb-Margalit'

- *Lecture 5*: More on the complex analytic side: divisors and the Riemann-Roch Theorem, with applications including Uniformization for genus 0 surfaces, and embedding in projective space.
  References: §5.4 and 5.7 of Jost's book.

- *Lecture 6:* Harmonic maps , Wolf's parametrization of Teichmüller space as an example of the interaction between hyperbolic geometry and complex analysis.
  References: Chapter 3 of Jost's book, and M. Wolf's paper "Teichmüller theory of Harmonic Maps" (Journal of Differential Geometry, 1989). For a proof of the uniformization theorem see §4.4 of Jost's book.

As mentioned before, the two perspectives lead to a rich interplay of many parts of mathematics – we shall be selective and pick only a few aspects. In particular, we shall ignore Wolpert's work which relates hyperbolic geomery and the symplectic geometry of the moduli spaces, or the work of Ahlfors-Bers (originating in ideas of Teichmüller) that used quasiconformal maps and analytical techniques to prove, for instance, the existence of a complex structure on Teichmüller space.