IISER PUNE LECTURES ON DIFFERENTIAL GEOMETRY (LECTURES 5 AND 6)

1. An introduction to the ideas behind the proof of Kodaira embedding - The $\bar{\partial}$ equation

For *every* pair of two points p, q in X, suppose we manage to find a holomorphic section s (that obviously depends on p and q) with a given first-order Taylor expansion, i.e., if we are given two vectors $u_1, u_2 \in L_p, L_q$ respectively and two vectors $\vec{v_1}, \vec{v_2} \in L_p \otimes T^*X, L_q \otimes T^*X$ respectively then we find a *global* holomorphic section s of L^k for a fixed but sufficienty large k such that $s(p) = u_1, s(q) = u_2$ and $\nabla s(p) = \vec{v_1}, \nabla s(q) = v_2$ where ∇ is the Chern connection of the metric h^k on L^k having positive curvature. (Actually, choose local trivialisations such that the Chern connection is d at p, q in these trivialisations.) Also, define a Kähler metric on the tangent bundle of X given by the curvature of h on L.

If such is the case, then I claim that we have enough number of sections to ensure that the Kodaira map $p \to [s_0(p) : s_1(p) : \ldots : s_N(p)]$ is actually an embedding. Indeed,

(1) *Well-definedness* : For every point p, if $u_1 \neq 0$, then $s(p) = u_1 \neq 0$. Thus the map makes sense (i.e. nothing gets mapped to the absurd $[0 : 0 : 0 \ldots]$).
(2) *Injectivity* : If $p \neq q$, and $u_1 = u_2$, if $s_i(p) = s_i(q)\ \forall\ i$, then $s(p) = s(q)$ – A contradiction.
(3) *The derivative is injective* : Suppose it is not so at a point p. Assume without loss of generality that $s_0(p) \neq 0$. So we are in a coordinate patch U_0 in $\mathbb{CP}^N$. Thus the map in local coordinates (after choosing coordinates z_j in X such that p is at the origin) That is, there exists a tangent vector $v \neq 0 \in T_pX$ such that

$$\sum_j \frac{\partial(s_i/s_0)}{\partial z_j} v_j = 0\ \forall\ i. \tag{1.1}$$

Now using the assumptions above, choose a section s such that $s(p) = 0$ and $\nabla s(p) = v s_0(p)$ where we chose a local trivialisation such that the Chern connection is d at p and complex coordinates z such that the Kähler metric ω at p is standard. (Thus we can pretend that $\vec{v}$ is a cotangent vector even though it is actually a tangent vector.)

Now we calculate

$$\sum_j \frac{\partial(s/s_0)}{\partial z_j} v_j = \sum_j v_j^2 > 0. \tag{1.2}$$

But

$$s = s_0 c_0 + \sum_i c_i s_i$$
$$\Rightarrow \sum_j \frac{\partial(s/s_0)}{\partial z_j} v_j = 0 + \sum_i c_i \sum_j \frac{\partial(s_i/s_0)}{\partial z_j} v_j = 0. \tag{1.3}$$

This is a contradiction.

Hence, our problem has now been reduced to finding a $k >> 0$ such that L^k admits holomorphic sections with specified first order Taylor expansions at any two given points.

Exercise 1.1. *Prove that there are* smooth *(but not necessarily holomorphic) globally defined sections having specified first order Taylor expansions that are holomorphic in small coordinate neighbourhoods of p and q and supported on slightly bigger coordinate neighbourhoods.*

So really, our problem is to find *holomorphic* sections that do the job given that we can find smooth sections doing them. Suppose $\tilde{s}$ is such a smooth section. Then $\eta = \bar{\partial}\tilde{s} \neq 0$ at some places on the manifold (if it were 0 everywhere, then $\tilde{s}$ is holomorphic and we are done). If we can magically solve the PDE $\bar{\partial}t = \eta$ *with* the restriction $t(p) = t(q) = dt(p) = dt(q) = 0$ then $s = \tilde{s} - t$ does the job!

So how does one solve $\bar{\partial}t = \eta$ for any η satisfying $\bar{\partial}\eta = 0$ with the restriction that t has 0 first order Taylor expansion at two given points ? This is a rather weird boundary condition. In PDE theory, it is more reasonable to require, say, $\displaystyle\int_X e^{-\phi}|t|_h^2 \text{vol} < \infty$ if $\int_X |\eta|^2 e^{-\phi}\text{vol} < \infty$ for a given "weight" function ϕ. Formally speaking, suppose we choose the weight function ϕ to behave like $\ln|z|^{2n+1}$ (where $2n$ is the real dimension of X) near p and q (where p, q are the origins of the coordinate systems) then the integral being finite *forces* t to vanish upto the first order.

In other words, we hope to be able to solve $\bar{\partial}t = \eta$ where $\bar{\partial}\eta = 0$ and η satisfies $\int_X |\eta|^2 e^{-\phi}\text{vol} < \infty$, such that the solution t satisfies $\displaystyle\int_X e^{-\phi}|t|_h^2\text{vol} < \infty$ for any smooth function ϕ. (Granted that $\ln|z|^k$ is not smooth, but it can be approximated by $\frac{k}{2}\ln(|z|^2 + \epsilon^2)$ which are smooth.) It is precisely to solve this PDE that we need positive and large curvature.

Where does curvature feature into solving a PDE ? The point is the following : A linear PDE like $\bar{\partial}t = \eta$ can be abstractly thought of as a problem in functional analysis. Namely, given a (closed, densely-defined) linear operator $T : H_1 \to H_2$ between two Hilbert spaces, for a given $v \in H_2$, find $u \in H_1$ (if possible) such that $Tu = v$. Suppose we can solve this problem. Then, taking inner product with $w \in Dom(T^\dagger)$ on both sides, we see that $\langle T^\dagger w, u\rangle = \langle w, v\rangle$. This means that $\langle w, v\rangle \leq \|T^\dagger w\|\|u\| \leq C\|T^\dagger w\|$. Conversely, it may be proven that if we can show $\|T^\dagger w\| \geq C\|w\|$, then indeed there exists a solution u to the problem described above with $\|u\| \leq C$ (we are denoting all arbitrary constants by C. They are not equal to each other!)

The quest to prove an estimate of the type $\|\bar{\partial}^\dagger w\|_{L^2} \geq C\|w\|_{L^2}$ is what brings curvature into the picture. More precisely, the operator $\bar{\partial}^\dagger w$ is a first order differential operator acting on $(0,1)$-forms w having a simple formula (it is akin to the divergence of a vector field). Now just like $\nabla.\nabla = \Delta$, we can form a Laplacian (called the Hodge-Dolbeault Laplacian) $\Delta_1 = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$. But there is another choice of a Laplacian (called the rough Laplacian) given by $\Delta_2 = \bar{\nabla}^*\bar{\nabla}$. These two Laplacians differ by a curvature term : $\Delta_1 = \Delta_2 + Curvature$. Thus $w\Delta_1 w = w\Delta_2 w + Curvature(w, w)$. Integrating-by-parts, it turns out that $\|\bar{\partial}w\|^2 + \|\bar{\partial}^\dagger w\|^2 = \|\bar{\nabla}w\|^2 + \int Curvature(w, w) \geq \int Curvature(w, w)$. Now the problem is set up such that $\bar{\partial}w = 0$. Thus, if the curvature is strictly positive, indeed the desired estimate holds. Since this the curvature of $e^{-\phi}h^k$, k has to be large enough so that the positive term dominates the potential negative contributions of the other terms.

Let us make this strategy slightly more rigorous in the special case of Riemann surfaces. The point is that when $g \geq 2$, it turns out that the holomorphic cotangent bundle $T^{1,0}X^*$ is ample (i.e. it admits a metric of positive curvature). In the case of $g = 0$, the tangent bundle $T^{1,0}X$ is itself ample.

In the $g = 1$ case, finding an ample line bundle requires a little bit more work (although in this case one can prove the embedding theorem directly using the Weierstrass ρ function). So suppose we are given a Riemann surface X with an ample line bundle (L, h). Given any two points $p, q \in X$, we will prove that there exists a holomorphic section s with given first order taylor expansion at p, q. Here is a theorem of Hörmander stated in this special case.

Theorem 1.2. *Suppose (X, ω) is a Riemann surface whose $T^{1,0}X$ is equipped with a hermitian metric whose associated 2-form is ω, and $(L, \tilde{h})$ is a hermitian holomorphic line bundle. Suppose the curvature of ω is the 2-form Θ_ω (the Gaussian curvature essentially) and that of h is Θ_h. Let η be a smooth $(0,1)$-form that is $\bar{\partial}$-closed, i.e., $\bar{\partial}\eta = 0$. Assume that $\frac{\sqrt{-1}}{2}\Theta_\omega + \frac{\sqrt{-1}}{2}\Theta_h \geq c\omega$ where $c > 0$. Then there exists a smooth section t of L satisfying $\bar{\partial}t = \eta$ and*

$$\int_X |t|^2_{\tilde{h}}\omega \leq \frac{1}{c}\int_X |\eta|^2_h \omega \tag{1.4}$$

Remark 1.3. The same statement as above holds on higher dimensional complex manifolds (except that the curvature term coming from the manifold's Kähler metric is the Ricci curvature).

We shall not prove this theorem. But we shall indicate how this implies the Kodaira embedding theorem (for Riemann surfaces). Firstly, here is a simple exercise :

Exercise 1.4. *Prove that every metric on a Riemann surface is Kähler.*

Secondly, assuming theorem 1.2, let us see how to prove our main theorem - Kodaira embedding. Given two points (p, q) and coordinate systems z_p and z_q around p, q such that these points are the origin and such that the Kähler metric is standard upto the first order in these coordinates, and first order taylor expansions (in a trivilisation where the metric on the line bundle is standard upto the first order) $u_1 + z_p v_1$, $u_2 + z_q v_2$, let $\phi_\epsilon = \frac{3}{2}\rho_1 \ln(|z_p|^2 + \epsilon^2) + \frac{3}{2}\rho_2(|z_q|^2)\ln(|z_q|^2 + \epsilon^2)$ be a smooth function on X where $\rho_1 = \rho(|z_p|^2)$, $\rho_2 = \rho(|z_q|^2)$, and $\rho : \mathbb{R} \to \mathbb{R}$ is a smooth non-negative function whose support is in $(-\frac{1}{2}, \frac{1}{2})$ and is equal to 1 on $[-\frac{1}{4}, \frac{1}{4}]$. Let $\tilde{h}_\epsilon = h^k e^{-\phi}$ be a new metric on L^k. Its curvature (multiplied by $\frac{\sqrt{-1}}{2}$) is

$$\frac{\sqrt{-1}}{2}\Theta_{\tilde{h}_\epsilon} = k\frac{\sqrt{-1}}{2}\Theta_h + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi_\epsilon$$

$$= k\Theta_h + \ln(|z_p|^2 + \epsilon^2)\frac{3\sqrt{-1}}{4}\partial\bar{\partial}\rho_1 + \ln(|z_q|^2 + \epsilon^2)\frac{3\sqrt{-1}}{4}\partial\bar{\partial}\rho_2 + \frac{3\sqrt{-1}}{4}\partial\rho_1\frac{z_p d\bar{z}_p}{|z_p|^2 + \epsilon^2} + \frac{3\sqrt{-1}}{4}\frac{\bar{z}_p dz_p}{|z_p|^2 + \epsilon^2}\bar{\partial}\rho_1$$

$$+\frac{3\sqrt{-1}}{4}\partial\rho_2\frac{z_q d\bar{z}_q}{|z_q|^2 + \epsilon^2} + \frac{3\sqrt{-1}}{4}\frac{\bar{z}_q dz_q}{|z_q|^2 + \epsilon^2}\bar{\partial}\rho_2 + \frac{3\sqrt{-1}}{4}\rho_1\frac{\epsilon^2 dz_p d\bar{z}_p}{(|z_p|^2 + \epsilon^2)^2} + \frac{3\sqrt{-1}}{4}\rho_2\frac{\epsilon^2 dz_q d\bar{z}_q}{(|z_q|^2 + \epsilon^2)^2}. \tag{1.5}$$

Noting that $d\rho_1 = d\rho_2 = 0$ in a small neighbourhood of p, q, we see that the above expression is $\geq c\omega$ for $c > 0$ (independent of ϵ) if k is large enough (but independent of ϵ).

Now by exercise 1.1 there exists a smooth section $\tilde{s}$ having the correct Taylor expansion and is holomorphic in small neighbourhoods of p, q. Thus we may assume that $\bar{\partial}\tilde{s} = \eta = 0$ on the small regions where $\rho_1 = 1$ and $\rho_2 = 1$. Thus $\int_X |\eta|^2_{\tilde{h}_\epsilon}\omega < C$ where C is independent of ϵ. By Hörmander's theorem, there exists a smooth section t_ϵ satisfying $\bar{\partial}t_\epsilon = \eta$ and

$$\int_X |t_\epsilon|^2_h \omega \leq \int_X |t_\epsilon|^2_h e^{-\phi_{\epsilon=0}}\omega \leq \int_X |t_\epsilon|^2_h e^{-\phi_\epsilon}\omega < C \tag{1.6}$$

where C is independent of ϵ. At this point some fancy PDE theory (you can mumble phrases like "elliptic regularity", "Sobolev embedding", and "Arzela-Ascoli") takes over to tell us that indeed there is a smooth solution $\bar{\partial}t = \eta$ satisfying

$$\int_X |t|_h^2 e^{-\phi_{\epsilon=0}} \omega < C \tag{1.7}$$

Exercise 1.5. *Prove that equation 1.7 implies that t vanishes to the first order at p and q.*

2. In general, what are smooth vector bundles and why should you care ?

There are two ways of coming up with examples of simple closed curves. One is to write it as the image of a path $\vec{\gamma} : [0, 1] \to \mathbb{R}^2$ with non-zero derivative and another is to write it as the zero locus of a function $F(x, y)$ such that $\nabla F \neq 0$ on the zero locus (i.e. $(\cos(2\pi t), \sin(2\pi t))$ and $x^2 + y^2 = 1$). A natural question is "Can every compact, connected, 1-dimensional submanifold of the plane be described in both of these ways ?" The answer is YES. (BTW if you require the "path" to be a complex analytic function on the upper half-plane, and the function F to be complex analytic in a sense, then this version is related to the Uniformisation theorem of Riemann surfaces.)

Exercise 2.1. *Prove that indeed every compact, connected 1-dimensional submanifold of the plane can be written as the image of a path with non-zero derivative everywhere, and also as the zero locus of a function whose gradient when restricted to the submanifold is non-zero. (Hint : Use the classification of 1-manifolds)*

Another question is "Is every k-dimensional closed submanifold of $\mathbb{R}^n$ the zero locus of $n - k$ functions whose derivatives are linearly independent when restricted to the submanifold ?" The answer is NO in general. Indeed, how would we hope to attack such a problem ? For one thing, if it is the zero locus of such functions $f_1, \ldots, f_{n-k}$, then there are $n - k$ independent vectors at each point on the submanifold that are perpendicular to it.

Exercise 2.2. *Using the above observation, come up with a counterexample. (Hint : Look at exercise 8)*

The above being said, we can surely find such functions *locally* (because there are local normal vector fields). The difficulty seems to be finding them *globally* because the local functions do not agree on the intersections of open sets. This seems to be intimately tied to the problem of finding $n - k$ independent global normal vector fields.

Exercise 2.3. *Prove that if the closed submanifold M has dimension $n - 1$ and has 1 independent normal vector field on it, then indeed there is 1 function defining it having non-zero gradient on M. (I do not know if this is the case if you replace 1 with k. There is a mathoverflow question on this, but no consensus.)*

If you think about the above question enough, you will find that while the local functions do not patch up to form global functions to $\mathbb{R}^{n-k}$, they do form global functions to a different manifold which we shall call "The total space of a vector bundle" (this particular vector bundle is called the normal bundle). (In fact, if you have seen vector bundles earlier, then know this : *Every* real vector bundle is secretly the normal bundle of *some* appropriately chosen submanifold of $\mathbb{R}^N$ for N large enough. So why should you care about vector bundles ? To solve the above problem.) Indeed, this "total space of a vector bundle" is locally of the form $U \times \mathbb{R}^{n-k}$ where U is an open set of the manifold M the local functions $\vec{f_U}(x)$ are such that $\vec{f_U}(x) = g_{UV}(x)\vec{f_V}(x)$ for some invertible $n - k \times n - k$ matrix of real-valued functions $g_{UV}(x)$ on $U \cap V$. These matrices satisfy $g_{VU} = g_{UV}^{-1}$ and $g_{UV}g_{VW}g_{WU} = I$. So,

Exercise 2.4. *Prove that the following topological space is in fact a smooth manifold*

$$V = \frac{\cup_\alpha U_\alpha \times \mathbb{R}^r}{(p, v_\alpha) \equiv (p, g_{\alpha\beta}(p) v_\beta)}$$

where U_α are an open cover of a smooth manifold M, and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ are smooth $r \times r$ invertible matrix-valued functions on $U_\alpha \cap U_\beta$ satisfying $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = I$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Also prove that there is a map $\pi : V \to M$ such that $\pi^{-1}(p)$ (the "fibre" of p) is a vector space of dimension r and that for every point p, there is an open neighbourhood N_p in M such that $\pi^{-1}(N_p)$ is diffeomorphic to $N_p \times \mathbb{R}^r$ (whilst being compatible with the projection map π) with the diffeomorphism being linear on the fibres.

The above manifold V is called a real vector bundle of rank r with transition functions $g_{\alpha\beta}$. If you have seen vector bundles earlier, you would've seen a seemingly different definition :

Definition 2.5. A real rank-r vector bundle V over a smooth manifold M is a smooth manifold equipped with a projection map $\pi : V \to M$ such that for every point $p \in M$, the inverse image (the "fibre") $\pi^{-1}(p)$ is a real vector space V_p of dimension r. Moreover, around every point $p \in M$ there is an open neighbourhood N_p such that $\pi^{-1}(N_p)$ is diffeomorphic to $N_p \times \mathbb{R}^r$ that respects the projection and such that the diffeomorphism being a linear isomorphism on the fibres. Such a diffeomorphism on N_p is called a *local trivialisation*.

We also need a notion of a map between two vector bundles (for a category, you need objects and morphisms) :

Definition 2.6. Suppose V and W are vector bundles over M with projections π_1 and π_2. Then a smooth function $f : V \to W$ is called a vector bundle morphism/map if $\pi_1 = \pi_2 \circ f$ and f is a linear map when restricted to the fibres.

If f has an inverse which is also a vector bundle map, then f is said to be an isomorphism of vector bundles. (That is, V and W are practically the same objects in disguise.)

Definition 2.7. A section $s : M \to V$ of a vector bundle is a smooth map such that $\pi \circ s = Id$.

Exercise 2.8. *Prove that the two definitions of vector bundles given above, agree. (First make this statement precise and then prove it.)*

How does one construct vector bundles ? Well, one way is to embed your manifold in $\mathbb{R}^N$ and take the normal bundle (i.e. the subset $N \subset M \times \mathbb{R}^N$ containing (p, v) such that v is perpendicular to the tangent vector at p of any curve lying on M, passing through p.

Exercise 2.9. *Prove that this is a vector bundle).*

As I said earlier, you can prove (using the Whitney embedding theorem) that *every* real vector bundle arises this way. But we need more concrete constructions. So here are ways to construct bundles from already existing ones V and W with transition functions g and h (how do you come up with already existing ones ? more on this later).

(1) V^* is the dual vector bundle whose fibres are the duals of V_p and whose transition functions are $(g_{\alpha\beta}^{-1})^T$.
(2) $V \oplus W$ is a vector bundle whose fibres are the direct sums of the vector spaces and whose transition functions are $\begin{bmatrix} g_{\alpha\beta} & 0 \\ 0 & h_\alpha \end{bmatrix}$ (the direct sum of the matrices).

(3) $V \otimes W$ is a vector bundle whose fibres are tensor products of the vector spaces and whose transition functions are the Kronecker product of the matrices g and h, i.e., $\begin{bmatrix} g^{11}_{\alpha\beta} h_{\alpha\beta} & g^{12}_{\alpha\beta} h_{\alpha\beta} & \ldots \\ g^{22}_{\alpha\beta} h_{\alpha\beta} & g^{22}_{\alpha\beta} h_{\alpha\beta} & \ldots \\ \vdots & \ddots & \ldots \end{bmatrix}$

(4) Suppose $f : N \to M$ is a smooth map then $f^*(V)$ (called the "pullback of V") is a vector bundle over N with the same fibres but with transition functions $f^* g_{\alpha\beta} = g_{\alpha\beta} \circ f$. (In fact, it is a non-trivial result that any map homotopic to f induces the same pullback on vector bundles. An even more non-trivial result is that every smooth vector bundle is the pullback of a standard vector bundle over a standard manifold called a Grassmannian.)

(5) $\det(V)$ is a line bundle whose transition functions are $\det(g_{\alpha\beta})$.

Here are some examples of vector bundles :

(1) Given any manifold, here is a stupid vector bundle of rank $r = M \times \mathbb{R}^r$. This is called a trivial bundle.

 Exercise 2.10. *Prove that a rank-r vector bundle is trivial if and only if there are r smooth sections s_i such that $s_i(p)$ are form a basis of the fibre V_p at every point $p \in M$.*

(2) Recall that a vector bundle of rank 1 is called a line bundle. We will deal mainly with line bundles later on. There are very nice ways to construct line bundles in algebraic geometry. But there is a cute elementary construction : The infinite Möbius strip can be thought of as a line bundle over a circle.

 Exercise 2.11. *Prove the Möbius strip is a line bundle over a circle. Is it at trivial line bundle ?*

(3) The Tangent bundle of a manifold is a nice example : Suppose $M \subset \mathbb{R}^N$, then the tangent bundle $TM \subset M \times \mathbb{R}^N$ given by (p, v) such that v is the tangent vector at p to a curve γ lying on M passing through p. Another way to define TM is as follows : Suppose U_α is an atlas on the n-dimensional manifold M with transition functions $x_\alpha = \psi_{\alpha\beta}(x_\beta)$. Then TM is given by local trivialisations of the form $U_\alpha \times \mathbb{R}^n$ with transition functions $g^{ij}_{\alpha\beta} = \frac{\partial x^i_\alpha}{\partial x^j_\beta}$.

Lastly, the definitions above also make sense if you replace $\mathbb{R}$ with $\mathbb{C}$. Such bundles are called complex vector bundles. We will deal largely complex vector bundles. (How do you construct complex vector bundles ? One potential way is to take a real vector bundle and "complexify" it, i.e., pretend that the transition functions are matrices of complex numbers instead of real. After all, real numbers are complex too (ask any politician for the converse).)

3. Connections

Recall that our main aim is to classify vector bundles. What can be the obstruction for a bundle to be trivial, i.e., to define globally linearly independent sections that form a basis for every fibre ? One naive way would be to take sections that are "constant" (do not "vary"). In $\mathbb{R}^n$ what is the meaning of a constant function f ? Simply that its directional derivative $df(X) = 0$ everywhere and for all vectors X.

So it seems that defining the notion of a directional derivative of a section s of a bundle along a vector field X, namely $\nabla_X s$, is a useful thing to do. Whatever $\nabla_X s$ is, it at least ought to be a section of the bundle. Naively, how can one hope to do this ? After all, $s|_{U_\alpha} = \vec{s}_\alpha = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \ldots s_r \end{bmatrix}$ where U_α is an open set over which the bundle is trivial. So one could naively define $\nabla_X s = \begin{bmatrix} ds_1(X) \\ ds_2(X) \\ \ldots \end{bmatrix}$. The problem with this definition is that when you change your trivialisation to $\vec{s_\beta} = g_{\beta\alpha}\vec{s_\alpha}$ then $\nabla_X s$ does not change like how a section is supposed to ! That is, $\nabla_X s_\beta = ds_\beta(X) = d(g_{\beta\alpha}s_\alpha)(X) = dg_{\beta\alpha}(X)s_\alpha + g_{\beta\alpha}ds_\alpha(X)$. So there is an extra dg term. In fact, this sort of a problem even arises when you calculate the good ol' directional derivative of a vector field in $\mathbb{R}^2$ in polar coordinates.

Exercise 3.1. *Suppose $\vec{Y} : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth vector field. Write $\nabla_X Y$ in polar coordinates.*

So the correct thing to do is to write $\nabla_X s_\alpha = ds_\alpha(X) + A_\alpha(X)$ where A_α is an $r \times r$ matrix of $1 - forms$ called the connection matrix. When you change your trivialisation,

Exercise 3.2. *Show that if $\nabla_X s_\alpha = g_{\alpha\beta}\nabla_X s_\beta$ then $A_\alpha = g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta}g_{\alpha\beta}^{-1}$.*

So the connection $\nabla = d + A$ provides a way to take directional derivatives. It is not obvious that there exists at least one connection for any given bundle. For a trivial bundle, there is an obvious choice of a connection, namely, $A_\alpha = 0$ for all α.

Exercise 3.3. *Show using a partition-of-unity argument that given a vector bundle E over X, there exists a smooth connection A on it.*

We will mainly care about line bundles (Sushmita will need general vector bundles). For a line bundle, the transition functions are $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$. The transformation rule for a connection is $A_\alpha = A_\beta - d \ln g_{\alpha\beta}$.

Some people like to define connections abstractly as follows :

Definition 3.4. Suppose E is a vector bundle over M, X, Y are smooth vector fields on M, and $Vect$ and $\Gamma(E)$ are the infinite-dimensional vector spaces of smooth vector fields on M and sections of E respectively, then a connection $\nabla : \Gamma(E) \times Vect \to \Gamma(E \otimes T^*M)$ is a first order differential operator (i.e. $\nabla s(p)$ depends only on the value of s at p and its first derivative at p) satisfying the following properties :

(1) Linearity in X : $\nabla_{aX+bY}s = a\nabla_X s + b\nabla_Y s$. (In other words, ∇s is actually a 1-form that can take in tangent vectors X and spit out numbers.)
(2) Leibniz rule : Suppose g is a smooth function, then $\nabla(gs) = dgs + g\nabla s$.

Exercise 3.5. *Prove that the two notions (the abstract one and the concrete one) of connections coincide.*

Lastly, a metric h on a complex vector bundle E is (as the name suggests) a way to take the dot product of vectors from E, i.e., locally it is a positive-definite hermitian (or symmetric if the bundle is real) $r \times r$ matrix h_α such that $g_{\alpha\beta}^\dagger h_\alpha g_{\alpha\beta} = h_\beta$ and $\langle s, t\rangle = s_\alpha^\dagger h_\alpha t_\alpha$ where our convention for the dot product in the complex setting is different from the usual mathematician convention. A connection ∇ is said to be "compatible" with the metric h if $d\langle s, t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle$. In terms of local trivialisations this means that if you choose an orthonormal local trivialisation, then A_α is skew-hermitian matrix of one-forms.

Exercise 3.6. *Prove that every vector bundle has a metric and a compatible connection.*

4. Curvature

As an analyst, first order differential operators are pain in the neck. So usually one tries to apply them twice to get a second order operator. (Compare the usual gradient to the Laplacian. Which do you think has a prettier theory ?)

So, if we try to define $\nabla^2 s$, it should, morally speaking, depend on two derivatives of s. However, shockingly enough, when defined correctly, it depends only on the value of s at a point rather than *any* derivatives of it ! In fact, it will turn out that $\nabla^2 s = Fs$ where F is locally a matrix of two-forms called the curvature of ∇.

Locally, $\nabla s_\alpha = ds_\alpha + A_\alpha s_\alpha = (d + A)s_\alpha$. Thus, define $\nabla^2 s_\alpha = (d + A) \wedge (d + A)s_\alpha = (d^2 + d \circ A + A \wedge d + A \wedge A)s = d(As) + A \wedge ds + A \wedge As = dAs - Ads + Ads + A \wedge As = (dA + A \wedge A)s$. Define $F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$. This transforms under change of trivialisation as

Exercise 4.1. *Prove that* $F_\alpha = g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1}$.

Finally, we specialise to the case of line bundles. For them, $F_\alpha = dA_\alpha$. Moreover, $F_\alpha = F_\beta = F$. So the curvature of a line bundle is a *globally* defined 2-form. Also, $dF = d^2 A = 0$. So it is a closed 2-form. In addition, if A_1, A_2 are two different connections, then $a = A_1 - A_2$ is actually a globally defined 1-form (Why?) Thus $F_2 = F_1 + da$. Moreover, if Σ is a 2-dimensional submanifold of M, then $\int_\Sigma F_2 = \int F_1 + da = \int F_1 + 0$ (by Stokes) is actually independent of the connection chosen. Actually it turns out that this number $\int_\Sigma F$ is always of the form $-2\pi\sqrt{-1}n$ where n is an integer.