IISER PUNE LECTURES ON DIFFERENTIAL GEOMETRY (LECTURES 3 AND 4)

1. How to potentially come up with an embedding

Suppose X is a compact complex manifold. If we want to find an embedding $f : X \to \mathbb{CP}^N$ then basically we want a map of the form $f(p) = [s_0(p) : s_1(p) : \ldots]$ where as we know, $s_1, s_2, \ldots$ are not really holomorphic functions of p (note that X_0, X_1 etc are sections of the $\mathcal{O}(1)$ line bundle) but are actually supposed to be interpreted as sections of a line bundle. So we turn this around and say, suppose L is a holomorphic line bundle over X.

Suppose $s_0, s_1, \ldots$ form a basis for the vector space of holomorphic sections. The first fact (which is actually reasonably deep) is that this space (let's call it $\Gamma(L)$) is finite dimensional. (Essentially, the point is that the vector space of solutions of certain kinds of PDE on compact manifolds is finite dimensional. Holomorphicity simply means that the Cauchy-Riemann equations are satisfied.) Suppose its dimension is $N + 1$. Choose a local trivialisation over U_α for L. The sections s_i are now holomorphic functions $s_{i,\alpha}$ on U_α. I claim that the map $p \to [s_{0,\alpha}(p) : s_{1,\alpha}(p) \ldots]$ makes sense, i.e., it does not depend on the choice of trivialisation. Indeed, suppose U_β is another trivialisation. Then on $U_\alpha \cap U_\beta$, $s_{i,\alpha} = g_{\alpha\beta} s_{i,\beta}$. Then $[s_{0,\alpha} : s_{1,\alpha} \ldots] = [g_{\alpha\beta} s_{0,\beta} : g_{\alpha\beta} s_{1,\beta} \ldots] = [s_{0,\beta} : s_{1,\beta} \ldots]$.

But the map above may still not be well-defined ! Indeed, what if there are no holomorphic sections of L ? What if there are a few sections, but all of them vanish at a point q ? Then the map is not well-defined there. So for the map to even be well-defined (forget about being an embedding) there must exist "enough" number of sections such that at every point p on X, there exists one section s_i such that $s_i(p) \neq 0$.

If the map is well-defined, for it to be an embedding, it must be

(1) Injective : Meaning that suppose p and q are two distinct points in X, there must be at least one holomorphic section s such that $s(p) \neq s(q)$ (i.e. "sections should separate points").
(2) Derivative should be injective.

So we can expect such an embedding into projective space only if there exists a holomorphic line bundle L on X having lots of sections. (Such line bundles are called "very ample" by algebraic geometers.) But this is too hard to check. So the Kodaira embedding theorem gives us a "differentio-geometric" criterion on L so that the map defined above is an embedding. The explanation of this differentio-geometric criterion will take some time. Let us first state the theorem.

Theorem 1.1. *The compact complex manifold X can be embedded into projective space if and only if there exists a holomorphic line bundle L that admits a metric having positive curvature (i.e., its Chern connection has positive curvature).*
Such bundles are called ample line bundles. Given an ample line bundle L, for all sufficiently large k, L^k (the tensor product of L with itself k times) is very ample, i.e., the map defined above is an embedding.

The above theorem states that if there is a line bundle satisfying some differentio-geometric requirement, then X is projective. But how the heck can one find such a bundle or prove that none exists ? That requires some more work. In particular, the so-called Lefschetz theorem on $(1, 1)$-forms helps (which is by the way, a special case of the Hodge conjecture). But we will not go into how one can apply this theorem. In what follows, we will firstly define what the "Chern connection" is, what it means for its curvature to be positive, give examples of holomorphic bundles (other than

$\mathcal{O}(1)$), and also give examples of the Chern connection. Then we will go on to the proof of Kodaira embedding (the basic idea is to reduce the problem to solving a PDE).

2. Preliminaries on the holomorphic tangent bundle, Chern connections on holomorphic line bundles, and curvature of line bundles

In the case of smooth manifolds, one has the concept of a tangent space T_pX at every point p. The set of these tangent spaces can be equipped with a natural manifold structure. (The point being - to define vector fields and construct diffeomorphisms of manifolds using vector fields.) This is done as follows : The vectors $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \ldots$ span T_pM at p and in fact, for nearby points where the coordianates x_i are defined. Suppose $v = \displaystyle\sum_i v_i \frac{\partial}{\partial x_i}$ is a tangent vector. If we change coordinates to y^i, then by the chain rule $v = \displaystyle\sum_i w_i \frac{\partial}{\partial y_i}$ where $w_i = \displaystyle\sum_j \frac{\partial y_i}{\partial x_j} v_j$. In other words, we can define an object akin to a line bundle (this time called a vector bundle) that equips the set of tangent spaces with a smooth manifold structure as follows : $TX = \frac{\cup_\alpha U_\alpha \times \mathbb{R}^n}{(p,\vec{v}_\alpha)\equiv(p,g_{\alpha\beta}\vec{v}_\beta)}$ where $(U_\alpha, x_{\alpha,i})$ forms an atlas for X, and the matrix-valued smooth functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{GL}(n, \mathbb{R})$ are defined as $[g_{\alpha\beta}]_{ij} = \frac{\partial y_i}{\partial x_j} s$.

In general, a smooth real vector bundle V of rank r on a smooth manifold X is defined as $V = \frac{\cup_\alpha U_\alpha \times \mathbb{R}^r}{(p,\vec{v}_\alpha)\equiv(p,g_{\alpha\beta}\vec{v}_\beta)}$ where $X = \cup_\alpha U_\alpha$ of open sets, and $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(\mathbb{R}, r)$ are smooth functions satisfying $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = Id$. One can replace $\mathbb{R}$ with $\mathbb{C}$ to get a smooth complex vector bundle. If the base manifold is complex, one can ask whether the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \cap GL(\mathbb{C}, r)$ are holomorphic. If so, then such a vector bundle is called a holomorphic vector bundle. The tangent bundle TX is an example of a smooth real vector bundle on X.

Exercise 2.1. *Prove that a smooth real vector bundle defined as above satisfies the following properties.*

(1) *It is a smooth manifold of dimension $n + r$.*
(2) *It has a projection map $\pi : V \rightarrow X$ such that for every $p \in X$, $\pi^{-1}(p)$ is a vector space of dimension r. (Called the fibre at p.)*
(3) *Prove that around every $p \in X$, there is a neighbourhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{R}^r$ with the diffeomorphism preserving the fibres and being linear on them. (This property is called being locally trivial.)*
(4) *Prove that any smooth manifold V satisfying the above properties is in fact diffeomorphic to a vector bundle as defined above, such that the diffeomorphism preserves fibres and is linear on them.*

If $s : X \rightarrow V$ is a smooth function satisfying $\pi \circ s(p) = p$, then it is called a smooth section (akin to the case of line bundles). If $s_1, \ldots, s_r$ are smooth sections on U which are linearly independent everywhere, then $V|_U$ is diffeomorphic to $U \times \mathbb{R}^r$ with the diffeomorphism preserving the fibres and being linear on them. In this situation, the s_i are said to be a local trivialisation.

Akin to line bundles, one can define the dual V^* of a vector bundle, the direct sum $V \oplus W$, the tensor product $V \otimes W$, and the pull-back f^*V. Moreover, one can define the vector bundles consisting of symmetric multilinear maps $Sym(V \times V \times \ldots V)$ and anti-symmetric multilinear maps $\Lambda(V \times \ldots V)$. When the latter construction is applied to the tangent bundle, one gets the bundle of differential forms.

How does one come up with examples of holomorphic vector bundles ? Here is a natural example : The so-called holomorphic tangent bundle $T^{1,0}X$. Indeed, just as the usual tangent spaces consist of tangent vectors of curves through a point, the holomorphic tangent space consists of complex tangents to complex analytic curves through a point. Just as $\frac{\partial}{\partial x^i}$ furnish a local trivialisation of the usual tangent bundle, $\frac{\partial}{\partial z_i}$ do the same for the holomorphic tangent bundle. More precisely, suppose U_α are complex coordinate charts on X (whose coordinates are $z_{\alpha,i}$, then $T^{1,0}X$ is locally trivial on U_α with transition functions $g_{\alpha\beta,ij} = \frac{\partial z_{\alpha,i}}{\partial z_{\beta,j}}$. It is not hard to prove that $T^{1,0}X$ is $\mathbb{R}$-linearly isomorphic to the real tangent bundle of X. There is an "anti-holomorphic" vector bundle called $T^{0,1}X$. I leave it to you to figure out what its definition ought to be.

The dual of $T^{1,0}X$ is denoted as $\Omega^{1,0}X$ and is locally trivialised by the $(1,0)$ forms $dz_1, dz_2 \ldots$. Likewise there is $\Omega^{0,1}X$. Now that we have 1-forms, we can build holomorphic bundles $\Omega^{p,0}X$ as the bundle of $(p,0)$ forms and more generally we get smooth (but not holomorphic) bundles of (p,q)-forms. The holomorphic bundle $\Omega^{n,0}X$ of top-forms is also denoted as K_X and is given a fancy name. It is called the "canonical bundle". In the case of Riemann surfaces, K_X is simply the dual of $T^{1,0}X$, i.e., it is the bundle of $(1,0)$-forms.

Just as we have the exterior derivative d that takes a k-form and spits out a $k+1$-form, we have two operators ∂ and $\bar{\partial}$ satisfying $\bar{\partial} f = \sum \frac{\partial f}{\partial \bar{z}^i} d\bar{z}^i$ for functions f and extended by the Leibniz rule to forms. Note that $d = \partial + \bar{\partial}$. Lastly, given a holomorphic vector bundle V, the operator $\bar{\partial}$ makes sense on sections of V. Indeed, suppose s is a section of a holomorphic vector bundle, then locally s is $\vec{s_\alpha}$, a vector full of complex-valued functions. Define the vector-valued $(0,1)$-form $\bar{\partial} s = \begin{bmatrix} \bar{\partial} s_{\alpha,1} \\ \bar{\partial} s_{\alpha,2} \vdots \end{bmatrix}$. My claim is that $\bar{\partial} s$ is actually a section of $V \otimes \Omega^{0,1}X$. Indeed, this can be seen easily by changing trivialisation and seeing how it transforms.

Now that we know that the Cauchy-Riemann operator $\bar{\partial} s$ makes sense, it is but natural to ask whether ∂s makes sense (so that ds would make sense). Unfortunately, suppose we choose two local trivialisations such that $s_\alpha = g_{\alpha\beta} s_\beta$, then $\partial s_\alpha = g_{\alpha\beta} \partial s_\beta + \partial g_{\alpha\beta} s_\beta$. However, not all is lost. While ds or ∂s do not make sense in general, one can "correct" them. A connection ∇ on a line bundle does precisely this. It is a first order differential operator (meaning that every point it depends on the first order taylor expansion of the section) such that ∇s is a section of $L \otimes TX$ (In simpler terms, given a vector field X, $\nabla_X s$ is supposed to be the "directional derivative" of s along X. Also, $\nabla_{\alpha X + \beta Y} = \alpha \nabla_X + \beta \nabla_Y$). It satisfies the Leibniz rule : $\nabla(fs) = df \otimes s + f \nabla s$ for any smooth function f.

Exercise 2.2. *Locally, prove that a connection ∇ on a smooth line bundle is $\nabla s_\alpha = ds_\alpha + A_\alpha s_\alpha$ where A_α is a 1-form such that under change of trivialisation, $A_\alpha = A_\beta - d \ln g_\alpha$.*

Given a metric h on a holomorphic line bundle L (i.e. a collection of positive functions h_α such that $h_\alpha = h_\beta |g_{\alpha\beta}|^2$, there is a very nice connection (called the Chern connection) that can be defined according to the formula $A_\alpha == \partial h_\alpha h_\alpha^{-1} = \partial \ln h_\alpha$.

Exercise 2.3. *Verify that the Chern connection is indeed a connection and that it is "compatible" with the metric, i.e., $d\langle s,t \rangle = \langle \nabla s, t \rangle + \langle s, \nabla t \rangle$.*

Exercise 2.4. *Suppose h is a metric on a holomorphic line bundle L. Around every point p let U be a coordinate chart on X with complex coordinates z_i. Assume also that L is locally trivial on U, i.e., every holomorphic section s of L on U can be identified with a holomorphic function s_α. Prove that*

(1) *We can find a holomorphic section s_α on U such that $|s|_h^2(p) = 1$. (Thus, if choose the trivialisation given by s, i.e., writing every vector w in the line bundle L at the point p as $w = vs(p)$ for a complex number v, then $|w|_h^2 = |v|^2$.*
(2) *Actually prove that we can find s such that $\frac{\partial s_\alpha}{\partial z_i}(p) = 0$. (In other words, in the trivialisation given by s, the metric is standard upto the first order.)*
(3) *Deduce that if we now change our trivialisation to the one given by s, then the Chern connection is* 0 *at p in this new trivialisation.*

The curvature Θ of a connection $\nabla = d + A$ on a smooth complex line bundle L is the 2-form $\Theta = dA$.

Exercise 2.5. *Verify that indeed Θ is well-defined as a global 2-form. Calculate it for the Chern connection. What happens to Θ when we change the connection ?*

As you can see, Θ is a closed 2-form such that if you change the connection, $\Theta_{new} = \Theta_{old} + da$ for some global 1-form a. Therefore, suppose Σ is a surface in your manifold X, the quantity $\displaystyle\int_\Sigma \frac{\sqrt{-1}\Theta}{2\pi}$ depends only on Σ and not on the connection used to calculate the curvature. In fact, this quantity is always an integer. The integrand is called the "first Chern class" of the bundle.

We say that the Chern connection associated to a metric h on a holomorphic line bundle has positive curvature if locally, in a holomorphic trivialisation, $\Theta = h_{i\bar{j}}dz^i \wedge d\bar{z}^j$ where $h_{i\bar{j}}$ is a Hermitian positive-definite matrix. A holomorphic line bundle L that admits a metric with positive curvature is called an "ample line bundle". As we saw earlier, the Kodaira embedding theorem implies that the presence of an ample line bundle on a compact complex manifold X *forces* X to be a submanifold of $\mathbb{CP}^N$ for some large N (which depends on how "ample" or how positively curved the bundle L, h is).

Exercise 2.6. *Prove that if (L, h) is positively curved, then the matrix $h_{i\bar{j}}$ defined as above defines a hermitian metric on the complex vector bundle $T^{1,0}X$, i.e., the quantity $\langle Y, Z\rangle = Y_\alpha^T h_\alpha \bar{Z}_\alpha$ is well-defined where Y, Z are complex tangent vectors lying in the span of $\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2} \ldots$.*

Exercise 2.7. *If L^* is the dual of a holomorphic line bundle L, and h is a metric on L, then prove that h^{-1} defines a metric on L^* and that its curvature is $-\Theta_h$. More generally, suppose $L^k = L \otimes L \ldots$ (k times) if $k > 0$ and then same with L^* if $k < 0$. Then define $h_{L^k} = h_L^k$. What is its curvature ?*

Remark 2.8. Given *any* Hermitian metric $h_{i\bar{j}}$ on the holomorphic tangent bundle $T^{1,0}X$ of a complex manifold, the 2-form $\omega = \frac{\sqrt{-1}}{2}h_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is well-defined as a 2-form. If this 2-form is closed, then the metric is said to be "Kähler". In such a case, this 2-form is called the "Kähler form" of h. So, a line bundle (L, h) being positively curved can also be stated as "The curvature form multiplied by $\frac{\sqrt{-1}}{2}$ is a Kähler form". However, not all Kähler forms arise as curvatures (indeed, if that were the case, every compact complex manifold would have been projective).

Exercise 2.9. *Suppose $h_{i\bar{j}}$ is a Hermitian metric on $T^{1,0}X$.*

(1) *Prove that $h_{i\bar{j}}$ induces a Riemannian metric g on the real tangent bundle TX via the $\mathbb{R}$-linear isomorphism between TX and $T^{1,0}X$.*
(2) *Conversely, given a g on the real tangent bundle that is "compatible with the complex structure" (phrase this rigorously using the proof of part 1 of this question), prove that induces a Hermitian metric h on $T^{1,0}X$.*

(3) *(Somewhat harder) Prove that h is Kähler (i.e. the associated 2-form ω is closed) if and only if there are local complex coordinates z^i such that $h_{i\bar{j}} = \delta_{i\bar{j}} + O(|z|^2)$, i.e., h is locally standard upto the first order. These are called "complex normal coordinates".*
(4) *Prove that the Riemannian volume form $\sqrt{\det(g)}dx^1 \wedge dx^2 \ldots$ is actually just $\frac{\omega^n}{n!}$.*

Here is an explicit example of a hermitian metric on $O(-1)$: Recall that the tautological line bundle $O(-1)$ is simply a subbundle of the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ such that the fibre over a point $[X_0 : X_1 : \ldots]$ consists of vectors $(v_0, \ldots, v_n)$ that point along $(X_0, X_1, \ldots)$. So there is an obvious metric on this. Indeed, suppose we choose charts U_i consisting of $X_i \neq 0$. Then $(v_0, \ldots, v_n) = \lambda_i(\frac{X_0}{X_i}, \frac{X_1}{X_i}, \ldots)$. Define $h_i = \displaystyle\sum_j \frac{|X_j|^2}{|X_i|^2}$.

Exercise 2.10. *Prove that indeed the above is an honest metric on $O(-1)$. Calculate its curvature. Is it positive ? What about the curvature of the dual metric on $O(1)$? What about $O(k)$?*

3. An introduction to the ideas behind the proof of Kodaira embedding - The $\bar{\partial}$ equation

For *every* pair of two points p, q in X, suppose we manage to find a holomorphic section s (that obviously depends on p and q) with a given first-order Taylor expansion, i.e., if we are given two vectors $u_1, u_2 \in L_p, L_q$ respectively and two vectors $\vec{v}_1, \vec{v}_2 \in L_p \otimes T^*X, L_q \otimes T^*X$ respectively then we find a *global* holomorphic section s of L^k for a fixed but sufficienty large k such that $s(p) = u_1, s(q) = u_2$ and $\nabla s(p) = \vec{v}_1, \nabla s(q) = v_2$ where ∇ is the Chern connection of the metric h^k on L^k having positive curvature. (Actually, choose local trivialisations using exercise 2.4 such that the Chern connection is d at p, q in these trivialisations.) Also, define a Kähler metric on the tangent bundle of X given by the curvature of h on L.

If such is the case, then I claim that we have enough number of sections to ensure that the Kodaira map $p \to [s_0(p) : s_1(p) : \ldots : s_N(p)]$ is actually an embedding. Indeed,

(1) *Well-definedness* : For every point p, if $u_1 \neq 0$, then $s(p) = u_1 \neq 0$. Thus the map makes sense (i.e. nothing gets mapped to the absurd $[0 : 0 : 0 \ldots]$).
(2) *Injectivity* : If $p \neq q$, and $u_1 = u_2$, if $s_i(p) = s_i(q)\ \forall\ i$, then $s(p) = s(q)$ – A contradiction.
(3) *The derivative is injective* : Suppose it is not so at a point p. Assume without loss of generality that $s_0(p) \neq 0$. So we are in a coordinate patch U_0 in $\mathbb{CP}^N$. Thus the map in local coordinates (after choosing coordinates z_j in X such that p is at the origin) That is, there exists a tangent vector $v \neq 0 \in T_pX$ such that

$$\sum_j \frac{\partial(s_i/s_0)}{\partial z_j} v_j = 0\ \forall\ i. \tag{3.1}$$

Now using the assumptions above, choose a section s such that $s(p) = 0$ and $\nabla s(p) = v s_0(p)$ where we chose a local trivialisation such that the Chern connection is d at p and complex coordinates z such that the Kähler metric ω at p is standard. (Thus we can pretend that $\vec{v}$ is a cotangent vector even though it is actually a tangent vector.)

Now we calculate

$$\sum_j \frac{\partial(s/s_0)}{\partial z_j} v_j = \sum_j v_j^2 > 0. \tag{3.2}$$

But

$$s = s_0 c_0 + \sum_i c_i s_i$$

$$\Rightarrow \sum_j \frac{\partial (s/s_0)}{\partial z_j} v_j = 0 + \sum_i c_i \sum_j \frac{\partial (s_i/s_0)}{\partial z_j} v_j = 0. \tag{3.3}$$

This is a contradiction.

Hence, our problem has now been reduced to finding a $k >> 0$ such that L^k admits holomorphic sections with specified first order Taylor expansions at any two given points.

Exercise 3.1. *Prove that there are* smooth *(but not necessarily holomorphic) globally defined sections having specified first order Taylor expansions that are holomorphic in small coordinate neighbourhoods of p and q and supported on slightly bigger coordinate neighbourhoods.*

So really, our problem is to find *holomorphic* sections that do the job given that we can find smooth sections doing them. Suppose $\tilde{s}$ is such a smooth section. Then $\eta = \bar{\partial}\tilde{s} \neq 0$ at some places on the manifold (if it were 0 everywhere, then $\tilde{s}$ is holomorphic and we are done). If we can magically solve the PDE $\bar{\partial} t = \eta$ *with* the restriction $t(p) = t(q) = dt(p) = dt(q) = 0$ then $s = \tilde{s} - t$ does the job!