IISER PUNE LECTURES ON DIFFERENTIAL GEOMETRY (LECTURES 1 AND 2)

1. What is the aim of this series of lectures ?

What is the aim of the study of manifolds ? (Differential topology) Perhaps to "write" a list of "standard" ones such that every manifold is diffeomorphic to one of these standard ones (classification). What about the study of Riemannian manifolds (M, g) ? (Riemannian geometry) To classify upto isometry. What about complex analytic geometry ? To classify upto biholomorphism. Algebraic geometry ? Upto birational equivalence.

But these are very hard problems. (In fact, for four manifolds and above, some of these problems are impossible to solve, i.e., there is no algorithm whose input is two 4 manifolds and whose output is yes if there are diffeomorphic and no otherwise.) A more reasonable question is : Can we come up with some standard manifolds such that every manifold is a submanifold of it ? (Yes - Whitney embedding theorem). What about the same question in other categories ? (Yes for Riemannian manifolds - The Nash embedding theorem, and complicated for compact complex manifolds - The Kodaira embedding theorem).

The main aim of this series of lectures is to define complex manifolds, give examples, and state the Kodaira embedding theorem. Along the way, if time permits, I want to do something that is perhaps only partially related to this main goal, namely, to define smooth vector bundles, connections, and curvature, and more importantly, convince you that the notion of a vector bundle is natural and quite central to Differential geometry. (Differential geometry is broadly speaking, a study of distances and angles using calculus. More precisely, it deals with Riemannian geometry, geometry of vector bundles, and symplectic geometry among other things.)

Prerequisites are a good rigorous understanding of multivariable calculus, linear algebra, complex analysis, and a first course on manifolds (ideally some De Rham cohomology too, but let's see about that). If we ever get to the proof of Kodaira embedding, I would need you to know a little bit of Hilbert spaces as well.

Before proceeding further, let me clarify one thing - Normally, whilst dealing with manifolds, one is taught that there are "charts", i.e., maps $\phi_\alpha : U_\alpha \subset M \to \mathbb{R}^n$ such that the transition functions $\psi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ are diffeomorphisms. From now onwards, I will omit the ϕ_α pretend that U_α is simply a copy of $\mathbb{R}^n$.

Some of the exercises (and perhaps even the material) in these notes may be repeated. Apologies in advance. The reason is that I could not make my mind as to whether I ought to do vector bundles in detail before complex geometry or after or at all in the first place.

2. Complex manifolds and several complex variables

Recall that a smooth manifold of dimension n is locally homeomorphic to an open set $U \subset \mathbb{R}^n$ such that the transition maps between two open sets are smooth diffeomorphisms. In the same vein, a complex manifold of complex dimension n is locally homeomorphic to an open set $U \subset \mathbb{C}^n$ such that the transition maps between two open sets are biholomorphisms. To make sense of this definition and to see why you would bother defining such an object will occupy us for some time. Then we will see that it is actually easy to produce noncompact examples of complex manifolds but much harder to come up with compact ones. This will naturally lead us to the Kodaira embedding theorem.

What does high school coordinate geometry deal with ? Things like the circle $x^2 + y^2 = r^2$ or parabolae $y = x^2$, etc. You calculate their tangents, the number of points of intersection of two of such objects, the number of parabolae passing through some given number of points, given an equation $ax^2+bxy+cy^2$, finding out when it represents a circle, an ellipse, a parabola, or a hyperbola, etc. In general asking questions of this sort for objects defined by a polynomial $f(x, y) = 0$ (and its higher dimensional generalisations, $f_1(x_1, x_2, ..., x_n) = f_2(x_1, ...) = .. = 0$) is the subject of "Real algebraic geometry". The problem with real numbers is that $x^2 + y^2 = 1$ is a circle but $x^2 + y^2 = -1$ is a joke.

So it is clear that while real algebraic geometry is what we should be studying (because it is the most natural thing after the usual things that Greeks and other ancient beings studied), it is too hard and the theory will not be pretty. Sometimes equations have solutions and sometimes they don't. So we extend our study to complex numbers. That is, we interpret $x^2 + y^2 = 1$ as an object in $\mathbb{C}^2$. It is no longer a circle (after all, if you have 3 free parameters to play with, you get a 3-dimensional manifold). But the above questions all make sense and are much more easily answered. Moreover,

Exercise 2.1. *Prove that $x^2 + y^2 = 1$ is a submanifold of $\mathbb{C}^2$.*

But it is not any old submanifold. Note that its transition functions are actually analytic functions. It is an example of a complex manifold of dimension 1 (Is it compact ?). So, since we have such an abundant supply of naturally occurring complex manifolds (defined by zeroes of polynomials in $\mathbb{C}^n$), it is a good idea to systematically study them.

Coming back to the definition of a complex manifold, we need to make sense of the phrase "The transition maps between two open sets are biholomorphisms". First of all, we know (hopefully!) what a holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$ is : It is locally a power series in z. Another way of defining it is $\frac{\partial}{\partial \bar{z}} f = 0$ where $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right)$. This is just a compact way of writing the well-known Cauchy Riemann equations $u_x = v_y, u_y = -v_x$. Another way is to simply say that f is complex differentiable on the open set U, i.e., $\lim_{h\to 0} \frac{f(z + h) - f(z)}{h} = \frac{\partial f}{\partial z}$ exists. It is a miraculous fact (owing to the Cauchy-Goursat integral formula) that all of these are equivalent. A biholomorphism is simply a bijection such that f and f^{-1} are holomorphic. Now how can one hope to make sense of "$f : U \subset \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic" ? One way is to demand that the partial derivatives $\frac{\partial}{\partial z_i} f$ exist. Another is to aask whether the function is locally a power series. Yet another is to demand that the function be C^1 and that $\frac{\partial}{\partial \bar{z}_i} f = 0$ for all i. Thanks to a theorem of Hartog, all of these are equivalent. (Note that this is very non-trivial. We are effectively saying that in the complex setting, "separately differentiable implies differentiable in the usual sense".)

Several complex variables is strange subject. Note that in one complex variable, the function $f(z) = 1/z$ is holomorphic everywhere except at $z = 0$. In more than one complex variable we have Hartog's phenomenon : If $f : \mathbb{C}^n - \{|z_i| \leq r \ \forall\ i\} \to \mathbb{C}^n$ is holomorphic, then it extends uniquely to a holomorphic function on all of $\mathbb{C}^n$ provided $n \geq 2$. The proof is quite nice : Define f on $|z_i| \leq r \ \forall\ i$ as $f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{|w|=r+1} \frac{f(w, z)dw}{w - z_1}$. Of course the resulting beast is holomorphic on $|z_i| < r$. The thing we have to check is whether it is a holomorphic extension of the original function or not. Note that of course this agrees with f on the open set $r + 1 > |z_i| > r \ \forall\ i \neq 1$. By the identity theorem we are done.

Another thing : Suppose you have a closed subset S of $\mathbb{C}^n$ defined by $f(z_1, \ldots, z_n) = 0$ where f is holomorphic. How do you know when S is a complex submanifold of complex dimension $n-1$? In real variables, you use the implicit function theorem, i.e., if $\nabla f \neq 0$ everywhere when $f=0$, then indeed it is a submanifold. It is worth pointing out that the same thing holds in complex variables, i.e., if $\frac{\partial f}{\partial z_i} \neq 0$ for some i, then locally you can solve for z_i holomorphically in terms of the other variables. This follows from the inverse function theorem of SCV which in turn follows from the usual inverse function theorem and the Cauchy Riemann equations.

Exercise 2.2. *Formulate and prove the inverse function theorem in SCV.*

Lastly, before we return to manifolds, just as we have differential forms $x_2 e^{x_1} dx_1 + 5dx_2, \sin(x_1x_2)dx_1 \wedge dx_2$ etc in $\mathbb{R}^n$, we have several kinds of differential forms in $\mathbb{C}^n$:$\bar{z}_2 dz_1 + 4d\bar{z}_2, |z_1|^2 dz_1 \wedge dz_2 + f(x_2, y_2) dz_3 \wedge dz_4, 3dz_1 \wedge d\bar{z}_2, ...$ A (p,q) form has p number of dzs and q number of $d\bar{z}$s. Moreover, just like we have $d\eta = \dfrac{\partial \eta_J}{\partial x_i} dx_i \wedge dx_{j_1} \wedge dx_{j_2} \ldots$, we have ∂ and $\bar{\partial}$. For example, $\partial(|z_1|^2 dz_2 + z_2\bar{z}_3 d\bar{z}_3) = \bar{z}_1 dz_1 \wedge dz_2 + \bar{z}_3 dz_2 \wedge d\bar{z}_3$.

Returning to complex manifolds, all the usual definitions in real variables, like immersion, submersion, and submanifold carry over. By the way, complex manifolds of complex dimension 1 (real dimension 2) are called Riemann surfaces. Using our implicit function theorem, we can produce lots of examples :

(1) $x^2 + y^2 = 1$ in $\mathbb{C}^2$
(2) $x^2 + y^2 + z^2 = 1, 2x + 3y + 4z = 0$ in $\mathbb{C}^3$.

I claim that all of these are noncompact. Perhaps in these special situations, you can directly see that they are noncompact. But actually, a much stronger statement holds : Every holomorphic function $f : M \to \mathbb{C}$ where M is a compact complex manifold is a constant. Indeed $u = Re(f)$ achieves its maximum at a point $p \in M$. Now locally, choosing coordinates $z_1, \ldots, z_n$ in a coordinate unit ball B around p, we have a harmonic function u (actually much stronger than merely harmonic) attaining its maximum in the interior of the ball. This contradicts the maximum principle unless u is a constant.

Exercise 2.3. *Assuming the above statement, prove that there are no compact complex submanifolds of* $\mathbb{C}^n$.

So a natural question is "How can one produce examples of compact complex manifolds ?" One way is to produce one example of such a manifold and hope to construct lots of submanifolds of it. But how does one produce even one example ? In the real situation, one constructs examples of manifolds using the quotient construction. This can be applied here too. Here are two examples :

(1) The complex torus : Suppose Λ is a maximal lattice in $\mathbb{C}^n$, i.e., it is of the form $n_1e_1 + n_2e_2 \ldots$ where n_i are integers and e_i form a basis. Then $\frac{\mathbb{C}^n}{\Lambda}$ is diffeomorphic to $S^1 \times S^1 \ldots S^1$ (2n-torus). It is also a complex manifold.

    Exercise 2.4. *Prove that it is a complex manifold.*

(2) Projective space : Consider the set of all lines in $\mathbb{C}^{n+1}$. My claim is that this can be made into a manifold, in fact, a compact complex manifold. Indeed, a straight line through the origin is given by a non-zero vector $\vec{v}$ pointing along it. That is, the set of all such lines is $\mathbb{CP}^n = \frac{\mathbb{C}^{n+1} - \vec{0}}{\vec{X} \equiv \lambda \vec{X},\ \lambda \in \mathbb{C}^*}$. A point on projective space is represented by an equivalence class denoted as $[X_0 : X_1 : X_2 \ldots : X_n]$ where X_i are coordinates in $\mathbb{C}^{n+1}$.

    Exercise 2.5. *Prove that it is a compact complex manifold.*

Now can we come up with submanifolds of projective space ? In the real case, we could simply take zero loci of smooth functions. But unfortunately, there are no holomorphic non-constant functions. So the only option is to take maps from $\mathbb{CP}^n$ to another complex manifold and take its zero locus. Which complex manifold should we consider and how can we construct such maps ? Fortunately, there is a naive way to produce submanifolds of $\mathbb{CP}^n$ coming from algebraic geometry. After all, how does one produce submanifolds of $\mathbb{C}^{n+1}$? One takes a few polynomials $f_i(X_0, X_1, \ldots, X_n)$ and sets them to zero. The issue is that if we take $f_i(\lambda X_0, \lambda X_1, \ldots)$ then we may not get the same points and hence this submanifold may not descend to the quotient. However, if $f_i(\lambda X_0, \lambda X_1, \ldots) = \lambda^k f_i(X_0, X_1, \ldots)$, i.e., if f_i is a *homogeneous* polynomial, then surely its zero locus is a well-defined set on $\mathbb{CP}^n$. (Indeed, it is a hard result due to Chow (which is generalised by Serre to what is now called GAGA) that every compact complex submanifold of $\mathbb{CP}^n$ arises this way.) Indeed, here are examples of such submanifolds :

(1) $\sum a_i X_i = 0$. This is called a hyperplane. That this is a submanifold is easy to see : If $X_j \neq 0$, then we may divide by X_j, consider coordinates $z_{i,j} = \frac{X_i}{X_j}$ and see that since this is a linear relation, we can solve for one of these in terms of the others in a holomorphic manner.
(2) More generally, if $F(X_0, \ldots, X_n) = 0$ is a homogeneous polynomial such that $\nabla F \neq 0$ on the zero locus, then this defines a submanifold of $\mathbb{CP}^n$. Indeed, suppose we choose a coordinate chart where $X_0 \neq 0$ and assume that $\frac{\partial F}{\partial X_j} \neq 0$. Defining $z_i = \frac{X_i}{X_0}$ we see that $f(z_1, \ldots, z_n) = F(1, z_1, \ldots, z_n) = 0$. Taking derivatives we get $\frac{\partial f}{\partial z_j} = \frac{\partial F}{\partial X_j} \neq 0$. Therefore, by the implicit function theorem, we are done. (Suppose $j = 0$, and that $\frac{\partial f}{\partial z_i} = 0$ for all other i. This situation is not possible. (Why ?))
(3) Likewise, $\sum X_i^2 = 0$, $\sum X_i = 0$ defines a complex codimension-2 submanifold. (This can be easily generalised to a bunch of homogeneous polynomials with independent derivatives.)

Now what are polynomials like $X_0, X_1^2 + X_2^2$ etc maps to ? They are surely not holomorphic functions on $\mathbb{CP}^n$. Let's write them down in local coordinates. Take the degree one polynomial X_1. Now suppose we choose a coordinate chart $U_0 : X_0 \neq 0$. Then $z_i = \frac{X_i}{X_0}$ are local coordinates on U_0, i.e., U_0 is homeomorphic to $\mathbb{C}^n$. Now the polynomial $X_1 = z_1 X_0$, i.e., it is "function" $z_1 : U_0 \to \mathbb{C}$. If we choose another coordinate chart like $U_j : X_j \neq 0$, then $X_1 = w_1 X_j$. Note that w_1 and z_1 are not the same on $U_j \cap U_0$ but are related by multiplication with $\frac{w_1}{z_1}$. Thus morally speaking, X_1 should be thought of as a map, not to $\mathbb{C}$ but to a manifold (which we shall denote as $\mathcal{O}(1)$) defined as $\frac{\cup_i U_i \times \mathbb{C}}{On\ U_i \cap U_j\ ,(p,v_i) \equiv (p, g_{ij} v_j)}$ where $g_{ij} = \frac{X_j}{X_i}$. These g_{ij} are obviously holomorphic functions from $U_i \cap U_j$ to $\mathbb{C}^*$. They also satisfy $g_{ij} = g_{ji}^{-1}$ and $g_{ij} g_{jk} g_{kl} = 1$. This manifold $\mathcal{O}(1)$ is known to algebraic geometers as the "Hyperplane line bundle on $\mathbb{CP}^n$". The g_{ij} are called the "transition functions" of the line bundle.

The manifold $\mathcal{O}(1)$ is a curious object. It admits an obvious "projection" map π to $\mathbb{CP}^n$ such that $\pi^{-1}(p)$ is $\mathbb{C}$, i.e., a 1-D complex vector space. So, in a sense, it consists of complex lines, varying holomorphically, parametrised by $\mathbb{CP}^n$. This is an example of a holomorphic line bundle. In general, a holomorphic line bundle L on a complex manifold X is simply a complex manifold $L = \frac{\cup_\alpha U_\alpha \times \mathbb{C}}{(p, v_\alpha) \equiv (p, g_{\alpha\beta} v_\beta)}$ where U_α is a collection of open sets on X such that $X = \cup_\alpha U_\alpha$ (they are called "trivialising open sets of L"), $g\alpha\beta : U_\alpha \cap U_\beta \to \mathbb{C}^*$ is a collection of holomorphic functions (called "the transition functions of L") satisfying $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$.

Exercise 2.6. *Prove that*

(1) *The holomorphic line bundle L is actually a complex manifold of dimension $n + 1$.*

(2) *Also prove that there is a holomorphic projection map $\pi : L \to X$ such that $\pi^{-1}(p) = \mathbb{C}$, i.e., a 1-D vector space. These vector spaces are called the "fibres" of the line bundle.*
(3) *Moreover, prove that around every point $p \in X$, there is an open set U such that $\pi^{-1}(U)$ is biholomorphic to $U \times \mathbb{C}$ with the map preserving fibres and the biholomorphism being linear on the fibres. (This is called being "locally trivial".)*
(4) *(Optional) Prove that every complex manifold L that satisfies the second and third points above is actually biholomorphic to the holomorphic line bundle we defined (with the biholomorphism preserving fibres and being linear on them).*

A holomorphic function $s : X \to L$ such that $\pi \circ s(p) = p$ is called a holomorphic section of L. For instance, $X_0, X_1, \ldots, X_n$ are holomorphic sections of $\mathcal{O}(1)$ over $\mathbb{CP}^n$. We say that $s : U \to \mathbb{C}$ provides a local trivialisation for L over U if $s \neq 0$ anywhere on U, i.e., using s one can show that L restricted to U is actually isomorphic to the trivial line bundle $U \times \mathbb{C}$.

Other than $\mathcal{O}(1)$ on $\mathbb{CP}^n$, what examples of holomorphic line bundles can we come up with ? A stupid example is $X \times \mathbb{C}$. This is (rightly) called the trivial line bundle over X. Here are some constructions of new line bundles from two given ones V and W on X with transition functions g and h.

(1) V^* is the dual line bundle whose fibres are the duals of V_p and whose transition functions are $g^{-1}_{\alpha\beta}$.
(2) $V \otimes W$ is a line bundle whose fibres are tensor products of the vector spaces and whose transition functions are the product of the matrices g and h.
(3) Suppose $f : N \to M$ is a holomorphic map then $f^*(V)$ (called the "pullback of V") is a vector bundle over N with the same fibres but with transition functions $f^*g_{\alpha\beta} = g_{\alpha\beta} \circ f$. For example, if $i : N \subset M$ is a complex submanifold of M, then $i^*(V)$ is called the restriction of V to N. The transition functions are simply restrictions.

The above definition of $\mathcal{O}(1)$ seems too contrived. Here is a more pleasant geometric definition of the Tautological line bundle $\mathcal{O}(-1)$.

Definition 2.7. The total space of the tautological line bundle is a subset of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ consisting of $([X_0 : X_1 : \ldots], v_0, v_1, \ldots, v_{n+1})$ such that $\vec{v} = \mu \vec{X}$ for some complex number μ. The projection map is $\pi([X_0 : X_1 \ldots], \vec{v}) = [X_0 : X_1 \ldots]$. In other words, on the space of lines through the origin, at every line, simply choose the 1-D vector space represented by that line. The dual bundle $\mathcal{O}(1)$ consists of linear functionals on each of those lines.

Exercise 2.8. *Prove that the tautological line bundle as defined above is indeed the dual of $\mathcal{O}(1)$ as defined earlier.*

Why is it denoted as $\mathcal{O}(1)$? The reason is that homogeneous polynomials of degree 1 are holomorphic sections of this bundle. (Indeed, we constructed this bundle so that precisely this happens.)

In fact, something stronger is true : All holomorphic sections of $\mathcal{O}(1)$ correspond to homogeneous degree-1 polynomials. (This is an example of the slogan of Serre's GAGA : "Analytic and algebraic geometry coincide on the projective space.")

Its proof is as follows :

Homogeneous degree-1 polynomials correspond to holomorphic sections of $\mathcal{O}(1)$: Indeed, given $F(X_0, X_1, \ldots) = \sum a_i X_i$ where at least one $a_j \neq 0$, we have already seen that these correspond to sections of $\mathcal{O}(1)$

(indeed $O(1)$ was defined in a such a way that X_1 corresponds to a section. You can easily verify that $\sum a_i X_i$ also canonically defines a section). However, we shall do this in another way, i.e., by interpreting $O(1)$ as the dual of the tautological line bundle $O(-1)$. A section of $O(1)$ is supposed to be a linear functional at every point $[X_0 : X_1 : X_2 \ldots]$ on the corresponding 1-D vector space consisting of vectors $\vec{v}$ lying along the line defined by $[X_0 : X_1 \ldots]$. In other words, define $\langle s_F([X_0 : X_1 \ldots]), \vec{v}\rangle = \sum a_i v_i$. This is holomorphic. Indeed, on $U_0 : X_0 \neq 0$ for instance (the other U_j behave similarly), $[X_0 : X_1 \ldots] = [1 : z_1 : z_2 \ldots]$ and $\vec{v} = v_0(1, z_1, z_2 \ldots)$, $\langle s_F(z_1, z_2, \ldots), v_0(1, z_1, \ldots) = v_0(a_1 + a_2 z_2 + \ldots)$. Thus, locally, $\langle s_F, \vec{v}\rangle$ behaves linearly in $\vec{v}$ and holomorphically in z as per definition.

All holomorphic sections correspond to homogeneous polynomials : Suppose s is a section of $O(1)$. Then at every point $[X_0 : X_1 : \ldots]$, $s([X_0 : X_1 \ldots])$ is a linear functional that takes $\vec{v} = \mu\vec{X}$ and spits out a complex number. This means that we can talk of a holomorphic function $F([X_0 : \ldots, X_n], v_0, v_1 \ldots) = \langle s([X_0 : X_1 \ldots]), v\rangle$ such that $F([X_0 : \ldots], \lambda v_0, \lambda v_1 \ldots) = \lambda F([X_0 : X_1 \ldots], \vec{v})$. Moreover, since $\vec{v} = \mu\vec{X}$, the previous function is actually simply a holomorphic function $F(X_0, X_1, \ldots, X_n)$ on $\mathbb{C}^{n+1} - \vec{0}$ such that $F(\lambda X_0, \lambda X_1, \ldots) = \lambda F(X_0, \ldots, X_n)$. By Hartog's theorem this extends to all of $\mathbb{C}^{n+1}$. Moreover, $\frac{\partial F}{\partial X_i}$ is a homogeneous function of degree 0. Thus it is a constant equal to its value at the origin. Thus F is linear. □

Exercise 2.9. *Define $O(k)$ as the tensor product of $O(1)$ with itself k-times. Prove that its holomorphic sections correspond to degree k homogeneous polynomials.*

The bottom line is that there are holomorphic line bundles on $\mathbb{CP}^n$ (and thus on its submanifolds) that have lots of holomorphic sections and that the "homogeneous coordinates" $X_0, X_1 \ldots$ on $\mathbb{CP}^n$ are secretly sections of a certain line bundle, namely, $O(1)$.

A natural question is "Which compact complex manifolds arise as submanifolds of $\mathbb{CP}^n$? (such things are called "projective varieties")" The answer to this question is provided by the Kodaira embedding theorem. As an application of the Kodaira embedding theorem, it turns out that if you choose a complex torus at random, then almost surely it will NOT be projective. On the other hand, all Riemann surfaces (1-D complex manifolds) are projective.