Miscellaneous Problems

(Concepts : Structure theorem for f.g. modules over PID, Vector spaces, Canonical forms)

Unless specified, R will denote a PID and F denotes the field $R/(p)$ for some prime p in R.

(1) State the structure theorem in all three forms: Invariant factor form, elementary divisor form, primary decomposition form

(2) (1) Let $M = R^r$. Then show that $M/pM \cong F^r$.
(2) Let $M = R/(a)$, where a is a non-zero element in R. Show that

$$M/pM \cong \begin{cases} F \text{ if } p \mid a \\ 0 \text{ if } p \nmid a \end{cases}$$

(3) State the uniqueness of the structure theorem.

(5) Suppose M is a Torsion module and

$$M \cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m)$$

where $a_i's$ are invariant factors. Show that $\text{ann}(M) = (a_m)$.

(6) Prove that if V is considered as a $F[X]$ module via a linear operator T then V is a direct sum of cyclic $F[X]$-modules. Further each cyclic module appearing in the sum is invariant under T.

(7) Show that if $M \cong R/(a_1) \oplus R/(a_2)$ where a_i's are non-zero, non-units and $a_1 \mid a_2$ then $ann(M) = (a_2)$

(8) Describe all possible 4×4 and 3×3 Jordan Canonical forms.

(9) Find $c_A(X)$, $m_A(X)$, invariant factors, Jordan form for the following matrices with coefficients in $\mathbb{R}$.

(a)

$$\begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -17 \\ 0 & 0 & 2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} -1 & 4 & 9 \\ 6 & -4 & -15 \\ -4 & 4 & 12 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(10) Find $c_A(X), m_A(X)$, invariant factors, rational canonical form for the following matrices

(a)

$$\begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -17 \\ 0 & 0 & 2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{pmatrix}$$

(11) Are the following matrices diagonalisable? Find the respective diagonal matrices to which they are similar.

(a)

$$\begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$