Assignment-4

(Concepts covered till now: Chain conditions, Primary decomposition in Noetherian rings, Spectrum of a ring and Zariski topology)

A will always denote a commutative ring with 1 and M will denote an A-module.

(1) Give an example of:

1. M which is an Artinian A-module but not Noetherian.
2. M which is a Noetherian A-module but not Artinian.
3. Subring of a Noetherian (Artinian) ring need not be Noetherian (Artinian)
4. M which is not Artinian A-module but every submodule of it is finitely generated.
5. primary ideal which is not a prime ideal.
6. ideal $\mathfrak{a} \in A$ such that $\mathfrak{r}(\mathfrak{a})$ is a prime ideal but $\mathfrak{a}$
7. a primary ideal in A which is not of the form $\mathfrak{p}^n$ where $\mathfrak{p}$ is a prime ideal.

(2) Show that if A is Noetherian (Artinian) ring then any finitely generated A-module is Noetherian (Artinian)

(3) Prove that a primary ideal of $\mathbb{Z}$ is either $\{0\}$ or $\mathfrak{p}^n$ of a prime ideal $\mathfrak{p}$.

(4) Prove that if $\mathfrak{a}$ is a proper ideal in a Noetherian ring then the prime ideals associated to $\mathfrak{a}$ are precisely the prime ideals which occur in the set of ideals $\{(\mathfrak{a} : x) \mid x \in A\}$.

(5) Show that in Spec($\mathbb{Z}$), only infinite set that is closed is Spec($\mathbb{Z}$).

(6) Describe what is Spec($\mathbb{Z}[X]$) and Spec($\mathbb{C}[X, Y]$).

(7) Show that the set $\{\mathrm{Spec}(\mathrm{A_f}) \mid \mathrm{f} \in \mathrm{A}\backslash\{0\}\}$ forms a basis for Zariski topology on Spec(A).

(8) Prove that Spec(A) is quasi-compact. (Note that in a Hausdorff space : quasi-compact = compact)

(9) Prove that Spec(A) is a T_0 space.