Assignment-3

(Concepts covered till now: Localisation, Integral extensions, Going up and Going down theorem)

Notation : A and B are commutative rings with identity and all modules under considerations are A-modules. $S \subset A$ is a multiplicatively closed subset

(1) Prove the following:

1. There exists a unique isomorphism $S^{-1}M \otimes_{S^{-1}A} S^{-1}N \to S^{-1}(M \otimes_A N)$ given by $\frac{m}{s} \otimes \frac{n}{t} \mapsto \frac{m \otimes n}{st}$
2. $S^{-1}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(S^{-1}\mathfrak{a})$

(2) Show that if $f : A \to B$ is a ring homomorphism and $\mathfrak{p} \subset A$ is a prime ideal then $\mathfrak{p}$ is a contraction ideal $\iff$ $\mathfrak{p} = \mathfrak{p}^{ec}$.

(3) Suppose $\phi : M \to N$ is a A-module homomorphism. The following are equivalent

1. ϕ is surjective
2. $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is surjective for all $\mathfrak{p} \in$ Spec(A).
3. $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is surjective for all $\mathfrak{m} \in$ max-Spec(A).

(4) If A is integrally closed domain, then show that $A[x_1, x_2, \cdots, x_n]$ is integrally closed.

(5) State the Going up theorem. Which extra hypotheses is required on rings A and B so that going down theorem will hold ? Give an example to show that these extra hypotheses are must otherwise the going down theorem need not be true.

(6) Prove that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$ is $\mathbb{Z}[i]$.

(7) Show that the following are equivalent:

1. M is a flat A- module
2. $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module for all prime ideals $\mathfrak{p} \subset A$.
3. $M_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m} \subset A$.

(8) Let $A := K[X^3, X^2Y, Y^3] \subseteq B := K[X, Y]$. Answer the following

1. Give one maximal ideal of A

2. What is integral closure of A in B? Is B/A an integral extension?
3. What is the field of fractions of A?
4. Is $X \in$ F.O.F(A)?
5. Give an example of an element of B that is integral over $\langle X^3, X^2Y, Y^3\rangle$.