Assignment-2

(Concepts covered till now: Functors : $\mathrm{Hom}(M, -)$, $\mathrm{Hom}(-, N)$, $M \otimes -$, $- \otimes N$, exactness properties, Flat modules)

Notation : A and B are commutative rings with identity and all modules under considerations are A-modules.

(1) Suppose $(R, \mathfrak{m})$ is a local ring. M and N are finitely generated A-modules with $M \otimes_A N = 0$. Prove that either $M = 0$ or $N = 0$. (Hint: Nakayama Lemma)

(2) Show that there exists unique module isomorphisms:
(a) $M \otimes_A (N \otimes_A P) \to (M \otimes_A N) \otimes_A P \to M \otimes_A N \otimes_A P$
(b) $(M \oplus N) \otimes P \to (M \otimes P) \oplus (N \otimes P)$.

(3) Show that for a fixed module N, the functor $- \otimes N$ is a left adjoint of the functor $\mathrm{Hom}(N, -)$

(4) Show that in general:
(a) $- \otimes N$ is not a left exact functor.
(b) $\mathrm{Hom}(-, N)$ and $\mathrm{Hom}(M, -)$ are not right exact functors.

(5) Show that if M and N are both flat A-module then so is $M \otimes_A N$.

(6) Suppose that $M = \oplus_{i \in I} M_i$ is a direct sum of modules where I is an indexing set. Show that M is flat if and only if each M_i is flat.