Assignment-1

(Concepts covered : Radical, Extensions of ideals, Prime avoidance theorem, finitely generated modules, Exact sequences of Modules.)

Notation : A and B are commutative rings with identity and all modules under considerations are A-modules.

(1) Suppose that G is a p-group, p - a prime no. Show that the Jacobson radical $\mathfrak{J}(\mathbb{F}_p[G])$ is same as the nil-radical $\mathfrak{n}(\mathbb{F}_p[G])$

(2) Suppose $f : A \to B$ is a ring homomorphism. Prove that w.r.t. f, for an ideal $\mathfrak{a} \subset A$, $\mathfrak{a}^{ece} = \mathfrak{a}^e$.

(3) Suppose M is a f.g. module, $\mathfrak{a}$ is an ideal contained in the Jacobson radical $\mathfrak{J}(A)$, N is a submodules of M such that $M = \mathfrak{a}M + N$ then $M = N$.

(4) Prove that in the category of A-modules, given a short exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

the following are equivalent

1. $\exists$ a A-module morphism $e : M \to M'$ such that $e \circ f = Id_{M'}$
2. $\exists$ a A-module morphism $h : M'' \to M$ such that $g \circ h = Id_{M''}$
3. $M \cong M' \oplus M''$

Give an example where these conditions are not equivalent conditions.

(5) Prove that in the category of groups (4)(1) as above is equivalent to the statement that the middle term of the exact sequence is isomorphic to the direct product of the remaining two terms and (4)(2) equivalent to that statement that the middle term is the semidirect product $\mathrm{I}m(f) \rtimes \mathrm{I}m(h)$.

(6) Prove that a sequence $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact $\iff$ for all modules N, the sequence

$$0 \to \mathrm{Hom}(M'', N) \xrightarrow{g^*} \mathrm{Hom}(M, N) \xrightarrow{f^*} \mathrm{Hom}(M', N)$$

is exact.