SERC School on Nonlinear Dynamics
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Computer Lab Exercises
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You are encouraged to try out as many problems as possible. All the problems require the use of computers. You can use any programming language of your choice. They are classified into 3 different categories depending roughly on the level of difficulty. You can choose to do the problems in any order. They are placed approximately in an increasing level of difficulty. For most of these problems, you can refer to the book by Hilborn [1] or Strogatz [3] for additional help. Some online resources are suggested for numerical techniques at the end.

For Beginners

1. For the map $x_{n+1} = x_n^2$, with $0 \leq x_n \leq \infty$ try the following;
(a) Compute and plot x_n vs. n with $x_0 = 0.8$ and with $x_0 = 1.9$. If you wish, you can try with any other value of x_0 as well.
(b) Based on the plot obtained in (a), identify the fixed points for the map.
(c) What can be said about the stability of the fixed points based on the plotted results in (a).

2. For the differential equation $\dot{x} = \sin x$, do the following;
(a) What additional information is needed to obtain $x(t)$ numerically on the computer ?
(b) Once you figured out (a), use 2nd order Euler's method to compute the solution $x(t)$.
(c) Plot the computed solution and the exact analytical solution. Verify if they agree with one another.

3. Obtain numerical solutions to the system in problem (2) using the 2nd order Euler's method using step sizes $h = 0.1, 0.01, 0.001$. You can try other step sizes as well. What do you expect from computed solutions with these step sizes. Verify if the computed results are consistent with the error expected from Euler's method.

4. Obtain numerical solutions to the system in problem (2) using the 4th order Runge-Kutta method. Plot the numerical solutions obtained by Euler's method and RK method. Which of them is better ? How will you figure out the accuracy of these solutions ?

5. Generate 5000 random numbers using `ran()` or `rand()` (or any other such function call) provided in your system.
(a) Plot the random number sequence.
(b) Numerically compute and plot the distribution of random numbers. What do you infer from this plot ?
(c) Compute and plot the autocorrelation function ? What do you infer from this plot ?

6. Compute the machine epsilon for your computer in single and double precision. Machine epsilon is the smallest number when added to 1 gives a result different from 1. This is related to how the computers store floating point numbers. This exercise should be done using Fortran, C or C++ etc. In particular, do not use Matlab or Mathematica for this problem. Figure out why Matlab or Mathematica should not be used in this case ?

Intermediate

7. Consider the Logistic map

$$x_{n+1} = \mu x_n(1 - x_n)$$

(a) Compute and plot the trajectory x_n as a function of $n = 1, 2, 3......5000$ for several choices of μ.
(b) Compute and plot the bifurcation diagram as a function of the parameter μ.
(c) Compute and plot the invariant density for the logistic map at $\mu = 4$. Compare it with the analytical result for the invariant density.

8. Consider the logistic equation

$$\frac{dx}{dt} = \mu x(1 - x).$$

Solve this equation using Euler's or RK4 method for $\mu = 4$. Take $x(0) = 0.2$. Compare the trajectories of logistic equation and logistic map at $\mu = 4$.

9. Compute the Lyapunov exponent for the logistic map for various values of the parameter μ. See section 9.3 of Hilborn's book for one possible way of calculating the Lyapunov exponent. Compare the result with the bifurcation diagram in the same range of parameter.

10. Compute the solutions for tent map (with $\mu = 1.1, 2.0$) given by

$$\begin{aligned} x_{n+1} &= \mu x_n \qquad (x_n < 1/2) \\ &= \mu(1 - x_n) \quad (x_n \geq 1/2). \end{aligned}$$

Use the procedure in question 8 (from Ref. [1]) to compute the Lyapunov exponent for tent map and circle map.

11. For the logistic map in problem 1, compute x_n as a function of $n = 1, 2, 3......10000$ at $\mu = 2.0, 3.25, 4.0$. Compute and plot the power spectrum of x_n. What can you infer from the power spectrum and how does it correlate with the bifurcation diagram and the qualitative dynamics of logistic map.

12. For the system given by,

$$\begin{aligned} \dot{x} &= \mu x + y - x^3 \\ \dot{y} &= -x + \mu y + 2y^3, \end{aligned}$$

where μ is a parameter, use numerics to show that $\mu = 0$ is a Hopf bifurcation and classify it as subcritical or supercritical ?

13. Compute the solutions of

$$\ddot{x} + \omega^2 x = 0$$

with $\omega = 2$ and initial condition $x(0) = 0$ and $\dot{x}(0) = 1$. (a) Plot $x(t)$ for $t \in [0, 4\pi]$.
(b) Plot the phase space for this system.
(c) Plot $E = \dot{x}^2 + \omega^2 x^2$ as a function of t and verify that it is a conserved quantity.

14. Do the problem (12) in single and double precision and see its effect on how well E is conserved as a function of time. By using any algorithm of your choice, control the growth of error such that E is conserved correct to at least 6 decimal places.

15. Consider the Lorenz system with parameters (σ, r, b) ;

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz\end{aligned}$$

(a) Obtain the Jacobian matrix by linearising around the fixed point (x^*, y^*, z^*).
(b) Numerically compute the eigenvalues of the Jacobian matrix for the case with $(x^* = \sqrt{b(r-1)}, y^* = \sqrt{b(r-1)}, z^* = r-1)$. Take the parameter set to be $(10, 35, 2.666)$.
(c) What do you infer about the stability of Lorenz system in the vicinity of this fixed point.

<u>Advanced</u>

16. Sketch the bifurcation diagram, determine stability of the branches, detect and classify all bifurcations and find the bifurcation points of

$$\dot{x} = (r - x^2)(10r - x^4).$$

17. Compute the devil's staircase for the circle map

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi} \sin 2\pi\theta_n \qquad (\text{mod } 1)$$

for $K = 1$. See sections 6.7-6.8 in Hilborn's book [1] as well as the Devil's staricase shown in the figure below.

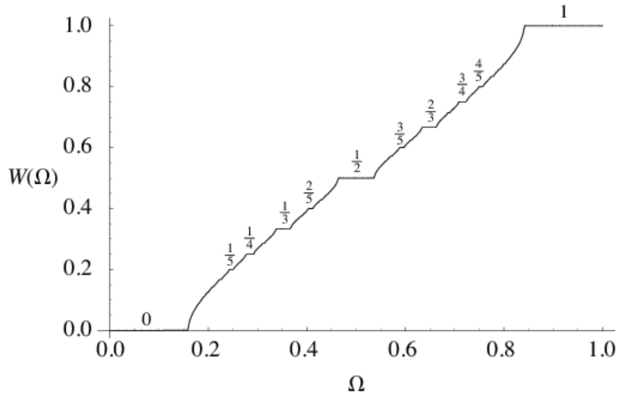

Figure 1 : Devil's staircase (winding number $W(\Omega)$ vs. Ω) for the circle map at $K = 1$.

18. For the Henon map given by

$$\begin{aligned}x_{n+1} &= y_n + 1 - ax_n^2 \\ y_{n+1} &= bx_n\end{aligned}$$

with parameters $a = 1.4$ and $b = 0.3$, simulate and plot x_n against y_n. You should verify that you get the Henon attractor.

19. For the Duffing oscillator given by

$$\ddot{x} + \alpha\dot{x} + x + \beta x^3 = f \sin \omega t$$

compute and plot the trajectories for $f = 0.54214, 0.5421, 0.54$ with $\beta = 1, \omega = 1, \alpha = 0.5$. Compute the bifurcation diagram for $0.538 < f < 0.544$. See Ref [3] for more details on this system.

20. For the Rossler system

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned}$$

with a, b, c being the parameters. For $a = 0.4, b = 2, c = 4$, compute and plot the attractor. Take a look at the time series $z(t)$.

21.(a) Compute the solutions for the damped and driven nonlinear oscillator

$$\ddot{x} + \alpha\dot{x} + x = f \sin \omega t$$

for the set of parameters $\alpha = 0.1, \omega = 1, f = 0.1$. Compute and verify that you get a limit cycle. Use RK-4 method to solve the problem.
(b) For $\alpha = 1$ and $f = 0$, compute and verify that you get an attractor in phase space.

22. Consider the standard map given by

$$\begin{aligned} p_{n+1} &= p_n + \frac{K}{2\pi} \sin(2\pi x_n) \\ x_{n+1} &= x_n + p_{n+1}. \end{aligned}$$

Plot the stroboscopic section for this map for $K = 0.2, 0.5$ and $K = 0.97, 2.0$. Relate it to the results of KAM theorem. Identify the golden KAM tori in the Poincare section for $K \approx 0.97$.

23. Consider the model Hamiltonian

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{x^2 + y^2}{2} + \alpha x^2 y^2.$$

Compute and plot the $x - p_x$ Poincare section for this system with parameter $\alpha = 0.1, 0.2$. What are the qualitative features about the dynamics that you can infer from the Poincare section. Verify that the energy is the conserved quantity for this system.

24. Create a real symmetric matrix (of order 1000) with entries drawn from appropriate Gaussian distribution. Diagonalise this matrix and compute eigenvalue density and spacing distribution. Refer to text book by M. L. Mehta on *Random matrices.*

Challenge problems

25. Bouncing Ball Model [4] (Everson 1986): An imperfectly elastic ball bouncing on relatively massive harmonically vibrated table with amplitude A and frequency ω. The displacement of the ball above the table is given as

$$\frac{d^2y}{dt^2} = -g - A\omega^2 \cos \omega t.$$

Using the scaling $y = A$ and $t = 1/\omega$, we get $\frac{d^2y}{dt^2} = -\gamma - \cos t$, where $\gamma = g/A\omega^2$ is the dimensionless acceleration. When $y = 0$ (the ball hits the table), then

$$\dot{y}_{new} = -\beta \dot{y}_{old},$$

where β reflects the elasticity of the collision ($\beta = 1$ and $\beta = 0$ correspond to completely elastic and inelastic collisions, respectively). If the displacement of the table is assumed small relative to the displacement of the ball, this system of ODEs can be approximated by a 2D mapping,

$$v_{n+1} = \epsilon v_n + (1+\epsilon)(1+\sin\tau_n),$$

$$\tau_{n+1} = (\tau_n + Bv_{n+1}) \mod 2\pi,$$

where v_n and τ_n is the velocity and time at the nth impact. Plot the bifurcation diagram for above system and analyze the transition points (assume $\epsilon = 0.3$).

27. Ueda's circuit [6] (Ueda 1979): Duffing's equation was studied by Ueda, exploring the dynamics of the transition of chaotic attractors (Ueda 1979). The system can be described by the second order differential equation

$$\ddot{x} + k\dot{x} - x + x^3 + = -B\cos t,$$

with $k = .1$ and $B = 12$. This particular system describes a circuit, Ueda's circuit, with nonlinear inductance and a sinusoidal voltage source. Solve the above system and show the trajectories.

30. Use any method that you know of to compute the integral correct to at least 10 decimal places.

$$\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{1}{x}\left(\frac{\log x}{x}\right)\, dx$$

Tips : Check for convergence of the integral under the limit.
Note : This was part of the 100-digit, $100 challenge posed by SIAM and set by Nick Trefethen in 2002. To see the other problems, go to http://people.maths.ox.ac.uk/trefethen/hundred.html .

Some useful online resources for numerical techniques :
[1] `http://www.mathworks.in/moler/chapters.html`
[2] `http://math.fullerton.edu/mathews/n2003/NumericalUndergradMod.html`

References :

[1] *Chaos and Nonlinear Dynamics*, Robert C. Hilborn, (Oxford, 2000).
[2] *Nonlinear Dynamics: Integrability, Chaos and Patterns*, M. Lakshmanan and S. Sahadevan, (Springer, 2003).
[3] *Nonlinear Dynamics and Chaos*, S. Strogatz, (Westview Press, 2001).
[4] R. M. Everson, Chaotic dynamics of a bouncing ball, *Physica D*, **19**, 1986, 355-383
[5] Kathleen T. Alligood, Tim D. Sauer, and James A. Yorke, Chaos, Springer, 1997
[6] Yoshisuke Ueda, Randomly transitional phenomena in the system governed by Duffing's equation, *Journal of Statistical Physics* **20**, 181-196 (1979).
[7] Drazin P. G. and Reid W. H., Hydrodynamic Stability, *Cambridge Mathematical Library*, 2004.