Black Body Radiation

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This notes is meant to be a supplement and not a substitute for standard text books.

1 Black body : A quick tour

Black body : An *idealised* model to study and understand the spectra of radiation emitted by a physical object or a body in thermal equilibrium maintained at temperature T. The ideal black body is a good emitter and a good absorber of radiation irrespecitve of its shape, size, colour or texture.

What is the problem : By 1870s, it was clear that light is part of a larger class called electromagnetic radiation (thanks to Maxwell's theory). The question arises as to what is the origin of electromagnetic radiation.

The pieces of black body jigsaw : Experimental black body results are shown in figure 1. The spectral energy density (energy content of black body per unit volume in the frequency range between ν and $\nu + d\nu$) is plotted as a function of frequency ν. How can it be understood from theoretical considerations.

What's the use : Many bodies in nature are close approximations to an ideal black body such as the Sun, earth and other heavenly bodies. The radiation from most objects on earth behaves as black body radiation. Understanding the black body radiation, among many other things, helps us to determine the temperature of the sun, other stars and planets, has implications for early universe and cosmology.

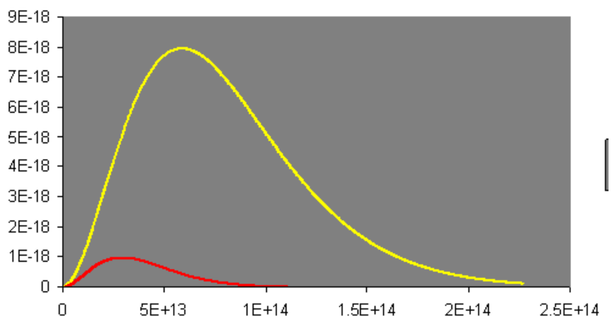

Figure 1: *The spectral energy density $u(\nu)\ d\nu$ plotted as a function of frequency for two different temperatures.*

2 The energy of a black body

For the purposes of calculations, it is convenient to think of black body as a cubic cavity of side L. A hole in one of the faces allows radiation to enter the cavity. Due to multiple reflections inside the cavity the radiation is nearly completely absorbed. Since the black body is in equilibrium at temperature T, the energy content of the black body is a constant except for small thermal fluctuations. This situation can be modelled by standing waves inside the black body cavity. Note that standing waves do not transmit any energy. This crucial idea is due to Lord Rayleigh who first attempted to explain the black body spectrum 1890s.

The total energy of the black body is stored in the form of standing waves. The technique is to count the number of standing waves with frequencies in the range ν and $\nu + d\nu$. Thus, we have,

$$E(\nu)\ d\nu = (\text{No.of standing waves})\ \bar{\epsilon}\ d\nu,$$

where, $\bar{\epsilon}$ is the average energy of each standing wave. Since we do not want our energy to be dependent on the volume of the cavity, we define energy density (energy per unit volume) in the frequency range ν and $\nu + d\nu$) to be,

$$u(\nu)\ d\nu = \frac{1}{L^3}(\text{No.of standing waves})\ \bar{\epsilon}\ d\nu \tag{1}$$

Now, the calculation boils down to determining the number of standing waves and the average energy of standing wave.

2.1 Number of standing waves

In principle, standing waves of all possible wavelengths should be present. However, the boundary conditions (waves should have a node at the walls of the cavity) of the cavity allow only modes of certain wavelengths to be present inside the cavity. The allowed wavelengths are obtained from the condition for standing waves in a cavity in one dimension to be $n = 2L/\lambda$, where λ is the wavelength of the standing wave and n is the number of half-wavelengths. In a 3D cavity, this condition is generalised to,

$$\begin{aligned} n_x &= 2L/\lambda, \qquad n_x = 1, 2, 3, ... \\ n_y &= 2L/\lambda, \qquad n_y = 1, 2, 3, ... \\ n_z &= 2L/\lambda, \qquad n_z = 1, 2, 3, ... \end{aligned}$$

In 3D, each triplet of integers (n_x, n_y, n_z) correspond to a possible mode of standing wave inside the cavity. In a cube of side L, evidently the largest allowed standing wave will have a wavelength L. This sets the upper limit for the allowed wavelengths or equivalently frequencies in the cavity. The number of standing waves above a given value of wavelength, say $\bar{\lambda}$, is the number of such triplets (or modes) which have wavelengths above $\bar{\lambda}$. There is an easier and approximate way to calculate this quantity. Consider a 3D space of integers (n_x, n_y, n_z) and every point in this space corresponds to one possible mode of standing wave. Since there are large number of modes, we can regard this space as being essentially continuous and ask how many independent modes lie in the range of wavelengths λ and $\lambda + d\lambda$. This is given by the surface area of a shell in one octant of sphere (see figure 2) as,

$$2\left(\frac{1}{8}\right)\ 4\pi n^2\ dn \tag{2}$$

The factor 2 comes from two possible states of polarisation for each standing wave. We want the result in terms of frequency and so we write n in terms of frequency as,

$$n = 2L/\lambda = \frac{2L\nu}{c} \qquad dn = \frac{2L}{c}d\nu \tag{3}$$

Substituting for n from Eq. 3 in Eq. 2 we get the result for number of standing waves in the cavity in $[\nu, \nu + d\nu]$ to be,

$$\text{No. of standing waves in } [\nu, \nu + d\nu] = \frac{8\pi L^3}{c^3}\nu^2\ d\nu \tag{4}$$

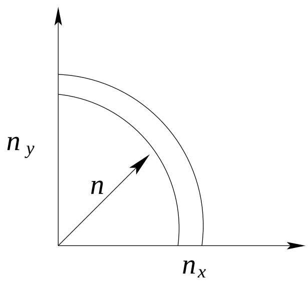

Figure 2: *Number of standing waves in the frequency range* $[\nu, \nu + d\nu]$*. We should determine the number of points in the shell of radius* n*. However, if the points are close enough we can simply assume them to be continuous and calculate the area of the shell.*

2.2 Average energy

The other ingredient we need to compute the energy density is the average energy of each mode of standing wave. Classically, this is obtained from the theorem of equipartition of energy which states that for systems in equilibrium at temperature T, the energy associated with each degree of freedom is $kT/2$, where k is the Boltzmann constant. Physically, the standing waves inside the cavity arise from harmonic oscillations of the electrons in the walls of the cavity. From the point of view of equipartition theorem, harmonic oscillator has two degrees of freedom (one potential and one kinetic) and hence the average energy is $\bar{\epsilon} = kT$.

3 Rayleigh-Jeans formula

Substituting all the known values in Eq. 1, we get the required energy density to be,

$$u(\nu)\ d\nu = \frac{8\pi\nu^2}{c^3} kT\ d\nu \tag{5}$$

This is the Rayleigh-Jeans formula. The obvious problem with this relation is that the total energy integrated from $\nu = 0$ to $\nu = \infty$ gives infinity. This is unphysical and clearly not supported by experimental results shown in fig 1.

4 Planck's radiation law

Max Planck assumed that the energy exchange between the oscillators on the walls of the cavity and the standing waves takes place in discrete quanta given by,

$$E_n = nh\nu \tag{6}$$

where h is the Planck's constant. Using this energy relation in Bolzmann distribution, $f(E) = \exp(-E/kT)$, he obtained the average energy to be,

$$\bar{\epsilon} = \frac{\nu}{e^{h\nu/kT} - 1} \tag{7}$$

Substituting this in Eq. 1, we get for energy density

$$u(\nu)\ d\nu = \frac{8\pi h}{c^3} \frac{\nu^3\ d\nu}{e^{h\nu/kT} - 1} \tag{8}$$

This is the Planck's radiation law and displays excellent agreement with the measured black body spectrum. In the limit, $\nu \to 0$, we recover the Rayleigh-Jeans law as in Eq. 5.

EXERCISE : *Calculate the average energy using Boltzmann distribution if the energy formula is $E = nh\nu$. The result is given in Eq. 7.*

5 Power radiated by a black body

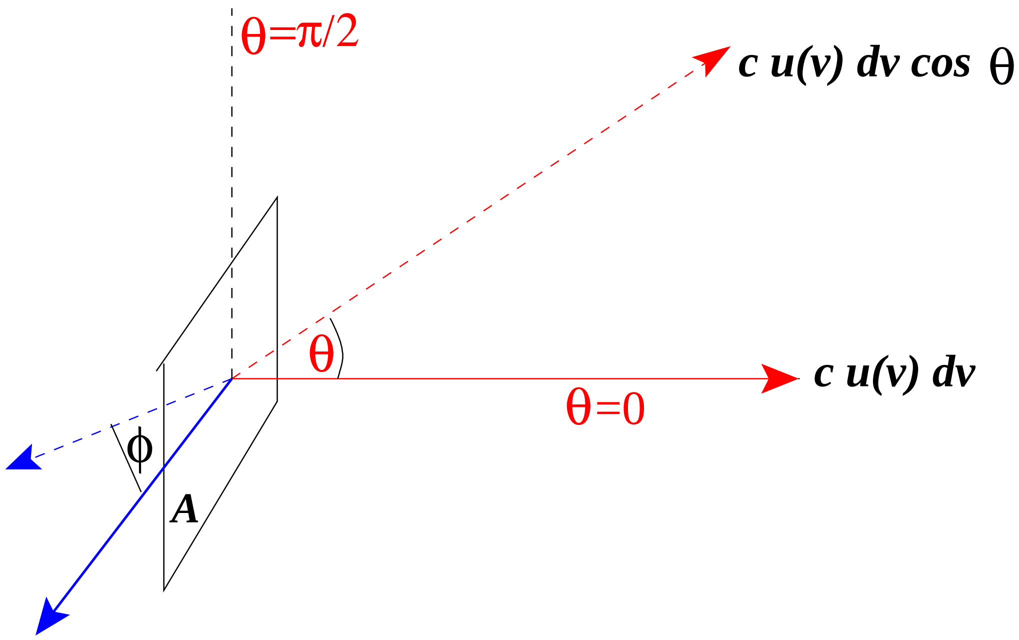

Figure 3: *The power emitted by face A of the imaginary cube will spread itself ouut in the entire hemishphere defined by $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$. The directions corresponding to $\theta = 0$ and $\theta = \pi/2$ are marked in the figure.*

Power, by definition, is the energy tranmitted per second. For the black body, we will calculate the power transferred by a black body per second per unit area. We will start from the energy density $u(\nu)\ d\nu$ in the frequency range $[\nu, \nu + d\nu]$. Consider a unit cube radiating outwards in all possible directions. Actually, as shown in Fig 3, we are interested in calculating the power radiated by the face A of the cube. Radiation emitted per unit area is the energy per unit volume (energy density) divided by the distance the radiation travels in time dt. It is given by,

$$u(\nu)\ d\nu\ c\ dt$$

Power radiated (energy transferred per unit time) per unit area is

$$u(\nu)\ d\nu\ c$$

This energy is radiated out in all possible directions, i.e, in all 4π solid angle. The power radiated in solid angle $d\Omega$ is given by,

$$u(\nu)\ d\nu\ c\ \frac{d\Omega}{4\pi}$$

Now, while doing this, irrespective of the shape of the black body, we are implicitly accounting only for the radiation coming out of unit surface area *in a given direction*. The radiation going out on other directions appears to be ignored in this calculation but would be accounted for when power radiated by the *entire* surface of black body is calculated.

Next, the radiation emerging from unit area of the black body surface will go out in all the directions in the hemisphere. Thus, the emerging radiation, in general, makes an angle θ with respect to the normal on the unit surface. Then, the power emitted in solid angle $d\Omega$ is,

$$u(\nu)\ d\nu\ c\ \frac{d\Omega}{4\pi}\ \cos\theta.$$

An area element for solid angle $d\Omega$ is given by,

$$d\Omega = \sin\theta\ d\theta\ d\phi, \qquad 0 \le \theta \le \pi/2, \qquad 0 \le \phi \le 2\pi.$$

Now, the power emitted in to a solid angle $d\Omega$ is given by,

$$u(\nu)\ d\nu\ \frac{c}{4\pi}\ \cos\theta \sin\theta\ d\theta\ d\phi$$

Now, the total power emitted per unit area of the black surface in the frequency range $[\nu, \nu + d\nu]$ is simply the integral of the power emitted in to the entire hemishphere. Thus, we get,

$$P(\nu)\ d\nu = u(\nu)\ d\nu\ \frac{c}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta\ d\theta\ d\phi \tag{9}$$

Substituting from Eq. 8 and doing the integrals, the final result is,

$$P(\nu)\ d\nu = \frac{2\pi h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1}\ d\nu. \tag{10}$$

This is the power radiated per unit area in frequency range $[\nu, \nu + d\nu]$. By integrating over ν, the total power radiated in all the frequencies can be obtained. This gives the well-known Stefan Boltzmann law.

EXERCISE : *Obtain an expression for Stefan-Boltzmann constant starting from Eq. 10.*

EXERCISE : *Calculate the power received by the planet Mars from the sun. And calculate the average temperature on the surface of Mars. Compare it with the measured temperature of Mars and explain the discrepancy, if any.*
