IISER, Pune

PHY441 / PH6414 : Quantum Information
Mid-semester Exam

Date : 30.9.2022 Time : 10 AM to 12 noon

Semester : Aug-Nov, 2022 Marks : 50 (maximum) Duration : 2 hours

Among questions 1 to 5, answer any four. Questions 6 to 8 are compulsory.
Show all the steps of your calculations clearly.
Use the same notation and symbols given in the questions.
Marks will be deducted if intermediate calculations are not shown.
If you are drawing graphs, label the axes.

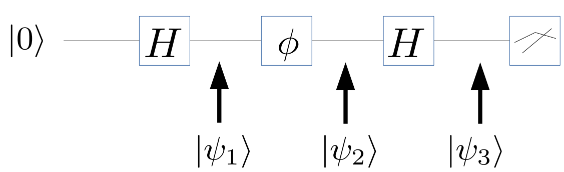
1. We measure the x-component of spin using projective measurement. If the initial state before measurement is $|1\rangle$, then use the projective measurement technique to find the probability of getting eigenvalue -1. What is the post-measurement state ? (5)

2. A density matrix is given as :

$$\rho = \begin{pmatrix} 2/3 & 3/4 \\ 1/4 & 1/3 \end{pmatrix}.$$

Find the expectation value of σ_y in the state represented by this ρ. (5)

3. Consider $|\psi_0\rangle$ as the input qubit state to the following circuit shown below :



In this, $|\phi\rangle$ represents a phase gate. Determine $|\psi_2\rangle$ and $|\psi_3\rangle$. Find the probability that the final qubit is $|1\rangle$. (5)

4. A measurement characterised by a set of positive operators $\{E_m\}$ such that $\sum_m E_m = 1$. If $E_m = M_m^\dagger M_m$ for some operator M_m, show that E_m is a Hermitian operator with non-negative eigenvalues. (5)

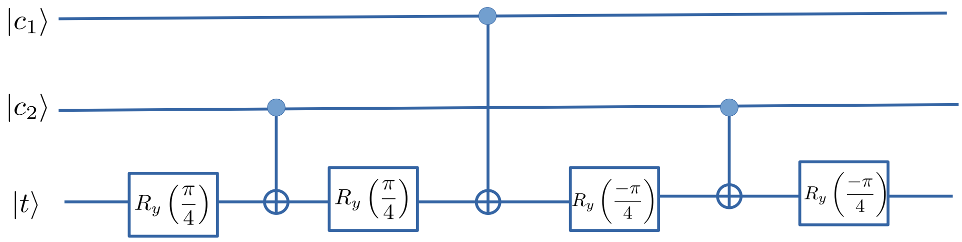
5. Prove the circuit identity : $U_{\text{CNOT}}\ (I \otimes Z)\ U_{\text{CNOT}} = (Z \otimes I)\ (I \otimes Z)$ (5)

6. If the σ_y Pauli matrix is

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

then, answer the following questions.
(a) Find the eigenvalues and eigenvectors of σ_y.
(b) Using spectral theorem, obtain an explicit matrix form of the operator e^{σ_y} in the computational basis. (4+6)

7. Consider the circuit shown below.



This circuit takes $|c_1, c_2, t\rangle$ to $e^{i\theta(c_1,c_2,t)}|c_1, c_2, t \oplus c_1 c_2\rangle$. Determine the phase $\theta(c_1, c_2, t)$. (8)

8. Consider the use of standard quantum teleportation protocol. Now, Alice wants to teleport a single qubit state $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$ to Bob who is located far away. Just before Alice measures her qubits, the joint state of Alice and Bob is given by,

$$|\psi_2\rangle = \frac{1}{2}\left(\, |00\alpha_1\rangle + |01\alpha_2\rangle + |10\alpha_3\rangle + |11\alpha_4\rangle \, \right).$$

Alice and Bob share a Bell state, *i.e*, an entangled pair $|\beta_{01}\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Answer the following questions.

(a) What are the states $|\alpha_1\rangle$ and $|\alpha_3\rangle$.

(b) If Alice sends two classical bits 1 and 0 to Bob, then what operators should Bob apply to recover the state $|\psi\rangle$.

(c) The entangled pair with Alice and Bob acquires an overall phase of the form $e^{i\phi/3}$. How does this affect the answer to part (b). Explain the reason for your answer. (4+4+4)