1)

Initial state before meas

$$|\psi_0\rangle = |1\rangle$$

$$\mathbb{P}_{-1} = |-\rangle\langle -|$$

$$P(-1) = \langle \psi_0 | \mathbb{P}_{-1} | \psi_0 \rangle$$

$$= \langle 1 | -\rangle\langle - | 1 \rangle$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{2} \langle 1 | (|0\rangle - |1\rangle)(\langle 0| - \langle 1|) | 1 \rangle$$

$$= \frac{1}{2} \rightarrow \text{prob of measuring } (-1)$$

Post-measurement

$$|\psi'\rangle = \frac{P_{-1} |\psi_0\rangle}{\sqrt{P(-1)}} = \frac{|-\rangle\langle -|1\rangle}{\sqrt{1/2}}$$

$$= -\frac{(|0\rangle - |1\rangle)}{\sqrt{2}} = -|-\rangle$$

2) $\langle \sigma_y \rangle = Tr(\rho \sigma_y)$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

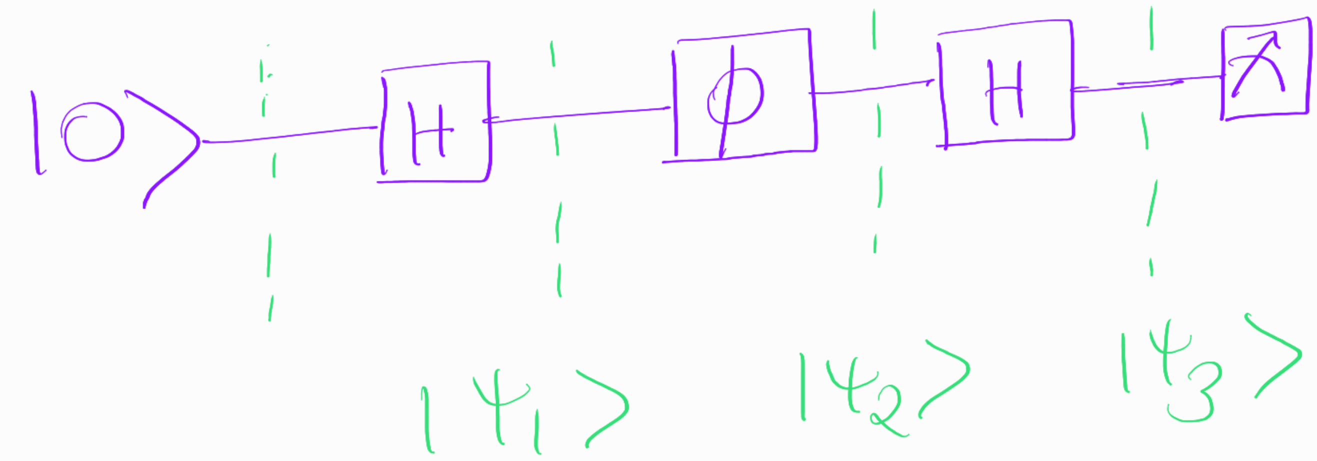
$$\rho = \begin{pmatrix} 2/3 & 3/4 \\ 1/4 & 1/3 \end{pmatrix}$$

$$\rho \sigma_y = \begin{pmatrix} 2/3 & 3/4 \\ 1/4 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4\, i & -2/3\, i \\ 1/3\, i & -1/4\, i \end{pmatrix}$$

$$Tr(\rho\sigma_y) = 3/4\, i - 1/4\, i$$

$$\langle \sigma_y \rangle = i/2$$

3) 

$$|\psi_1\rangle = H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|\psi_2\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} |\psi_1\rangle$$

$$= \frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = H|\psi_2\rangle = \frac{1}{2}\left(|0\rangle + |1\rangle + e^{i\phi}(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{2}\left((1+e^{i\phi})|0\rangle + (1-e^{i\phi})|1\rangle\right)$$

Prob of $|1\rangle$

$$= \left[\frac{1}{2}(1-e^{i\phi})\right]^2$$

$$= \frac{1}{4}(1-e^{i\phi})(1-e^{-i\phi})$$

$$= \frac{1}{4}(1-e^{-i\phi}-e^{i\phi}+1)$$

$$= \frac{1}{4}(2-2\cos\phi)$$

$$= \frac{1-\cos\phi}{2} = \sin^2\left(\frac{\phi}{2}\right)$$

4) $\sum_m E_m = 1$

$E_m = M_m^\dagger M_m$

E_m is hermitian

$\Rightarrow E_m^\dagger = E_m$

$E_m^\dagger = (M_m^\dagger M_m)^\dagger$

$= M_m^\dagger (M_m^\dagger)^\dagger$

$= M_m^\dagger M_m = E_m$

E_m is hermitian

$\Rightarrow$ all eigenvalues are Real

$E_m |\phi_k\rangle = m_k |\phi_k\rangle$

$\langle\psi|E_m|\psi\rangle \geq 0 \rightarrow$ positive op. for any $|\psi\rangle$

$\Rightarrow \langle\phi_k|E_m|\phi_k\rangle \geq 0$

$\Rightarrow m_k \geq 0.$

5)

$$U_{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

$$U_{CNOT}(I \otimes Z)$$

$$= (|0\rangle\langle 0| \otimes Z + |1\rangle\langle 1| \otimes XZ)$$

$$U_{CNOT}(I \otimes Z)U_{CNOT}$$

$$= |0\rangle\langle 0| \otimes Z + |1\rangle\langle 1| \otimes XZX$$

$$= |0\rangle\langle 0| \otimes Z - |1\rangle\langle 1| \otimes Z$$

$$= (Z \otimes I)(I \otimes Z)$$

6) $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

eigenvalues:

$+1$ $\qquad\qquad$ -1

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ $\qquad\qquad$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$|+i\rangle$ $\qquad\qquad$ $|-i\rangle$

By spectral thm:

$$\sigma_y = \sum_j \lambda_j |j\rangle\langle j|$$

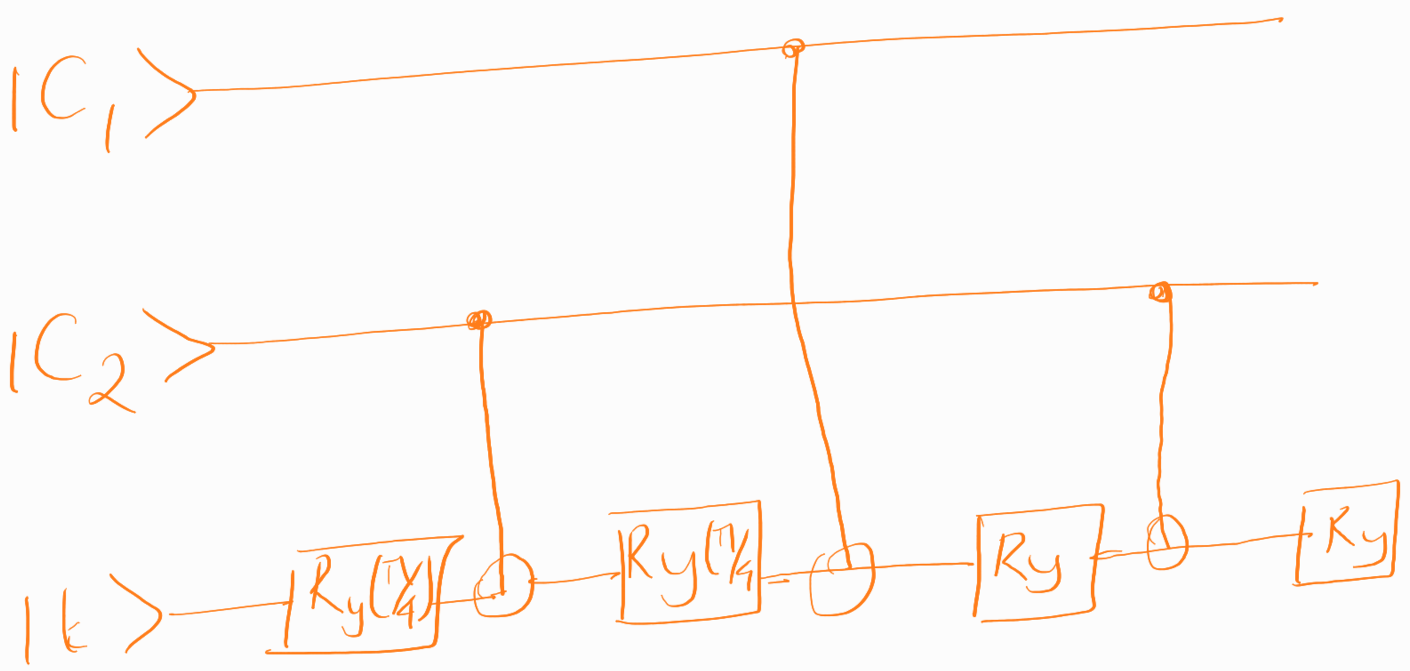
$$e^{\sigma_y} = e^{1}|+i\rangle\langle +i| + e^{-1}|-i\rangle\langle -i|$$

$$= \frac{1}{2}\left[e \begin{pmatrix}1\\ i\end{pmatrix}(1\ -i) + e^{-1}\begin{pmatrix}1\\ -i\end{pmatrix}(1\ i)\right.$$

$$= \frac{1}{2}\left[e \begin{pmatrix}1 & -i\\ i & +1\end{pmatrix} + e^{-1}\begin{pmatrix}1 & +i\\ -i & +1\end{pmatrix}\right]$$

$$= \begin{bmatrix}\cosh(1) & -i\sinh(1)\\ i\sinh(1) & +\cosh(1)\end{bmatrix}$$

7)



Case i)

$|c_1\rangle = 0, \; |c_2\rangle = |0\rangle$

$t_f \rightarrow$ final state

$$|t_f\rangle = R_y(-\pi/4)\, R_y(-\pi/4)\, R_y(\pi/4)\, R_y(\pi/4)\, |t\rangle$$

$$= |t\rangle$$

$$\Rightarrow \theta(0,0,t) = 0$$

ii) $|c_1\rangle = |0\rangle, \; |c_2\rangle = |1\rangle$

$$|t_f\rangle = R_y(-\pi/4) \times R_y(-\pi/4)\, R_y(\pi/4) \times R_y(\pi/4)\, |t\rangle$$

$$= |t\rangle$$

$$\theta(0,1,t) = 0$$

iii) $|c_1\rangle = |1\rangle, \; |c_2\rangle = |0\rangle$

$|t'\rangle =$

$R_y(-\pi/4)\, R_y(-\pi/4)\, X\, R_y(\pi/4)\, R_y(\pi/4)\, |t\rangle$

$= R_y(-\pi/2)\, X\, R_y(\pi/2)\, |t\rangle$

$R_y(\theta) = e^{-i\theta Y/2}$

$= \cos(\theta/2)\, I - i\sin(\theta/2)\, Y$

$R_y(\theta)\, X = X\, R_y(-\theta)$

$= X\, R_y(\pi/2)\, R_y(\pi/2)\, |t\rangle$

$= X\, R_y(\pi)\, |t\rangle$

$R_y(\pi) = -iY$

$= -iXY \Rightarrow Z$

$|t_f\rangle = Z|t\rangle$

$|t\rangle = |0\rangle$, $|t_f\rangle = |0\rangle \Rightarrow \theta(1,0,0) = 0$

$|t\rangle = |1\rangle$, $|t_f\rangle = -|1\rangle = e^{i\pi}|1\rangle$

$\Rightarrow \theta(1,0,1) = \pi$

iv) $|c_1\rangle = |1\rangle$, $|c_2\rangle = |1\rangle$

$|t'\rangle =$

$$R_y(-\pi/4) \times R_y(-\pi/4) \times R_y(\pi/4) \times R_y(\pi/4) \, |t\rangle$$

$$\Rightarrow \times R_y(\pi/4) R_y(-\pi/4) \underbrace{\times \times}_{I} R_y(-\pi/4) R_y(\pi/4) \, |t\rangle$$

$$\Rightarrow \times |t\rangle$$

$|t\rangle = |0\rangle$, $|t'\rangle = |1\rangle$, $\theta(1,1,0) = 0$

$|t\rangle = |1\rangle$, $|t'\rangle = |0\rangle$, $\theta(1,1,1) = 0$

$|1,0,1\rangle \rightarrow -|1,0,1\rangle \Rightarrow \theta = \pi$.

others $\theta = 0$

$$c_1(1-c_2)t\pi$$

$$|\psi_3\rangle = \frac{1}{2}\left(|00\alpha_1\rangle + |01\alpha_2\rangle + |10\alpha_3\rangle + |11\alpha_4\rangle\right)$$

$$|\psi_1\rangle = |\psi\rangle \otimes |\beta_{01}\rangle$$

$$= (a_0|0\rangle + a_1|1\rangle)\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(a_0|001\rangle + a_0|010\rangle + a_1|101\rangle + a_1|110\rangle\right)$$

$$U_{CNOT}|\psi_1\rangle = |\psi_2\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(a_0|001\rangle + a_0|010\rangle + a_1|111\rangle + a_1|100\rangle\right)$$

$$|\psi_3\rangle = \frac{1}{2}\left(a_0|001\rangle + a_0|101\rangle + a_0|010\rangle + a_0|110\rangle + a_1|011\rangle - a_1|111\rangle + a_1|000\rangle - a_1|100\rangle\right)$$

$$= \frac{1}{2}\Big(|00\rangle(a_0|1\rangle + a_1|0\rangle)$$
$$+ |01\rangle(a_0|0\rangle + a_1|1\rangle)$$
$$+ |10\rangle(a_0|1\rangle - a_1|0\rangle)$$
$$+ |11\rangle(a_0|0\rangle - a_1|1\rangle)$$

$\Rightarrow$

$|\alpha_1\rangle = a_0|1\rangle + a_1|0\rangle$

$|\alpha_3\rangle = a_0|1\rangle - a_1|0\rangle$

b) alice sends 10

$|\alpha_3\rangle = a_0|1\rangle - a_1|0\rangle$

$|\psi\rangle = a_0|0\rangle + a_1|1\rangle$

Bob operates

$ZX|\alpha_3\rangle = |\psi\rangle$

c) $|\beta'_{01}\rangle = \frac{e^{i\phi/3}}{\sqrt{2}}(|01\rangle + |10\rangle)$

$$|\psi_1'\rangle = \frac{e^{i\phi/3}}{\sqrt{2}}\left(a_0|001\rangle + a_0|010\rangle + a_1|101\rangle + a_1|110\rangle\right)$$

$$|\psi_2'\rangle = e^{i\phi/3}\,|\psi_2\rangle$$

$$|\psi_3'\rangle = \underbrace{e^{i\phi/3}}_{\text{global phase}}\,|\psi_3\rangle$$

Answer doesn't get affected.

8) $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$