Shift map

$$x_{n+1} = f(x_n) = 2x_n \pmod 1$$

This is just the doubling map.

Let's represent x in binary.

$$x = 0.\, b_1\, b_2\, b_3 \,\text{-----}$$

where each b_i represents 2^{-i} contribution to x.

Example:

$$\frac{1}{4} = 0 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 0 \times 2^{-4} + \cdots$$

$$= 0.01000\text{----} = 0.01\overline{0}$$

$$\frac{1}{5} = 0.00110011\text{-----} = 0.\overline{0011}$$

Iterate with some initial condition;

Let us take $x_0 = 0.0011\overline{0011} = 1/5$

$$x_1 = f(x_0) = 0.011\overline{0011}$$

$$x_2 = f^2(x_0) = 0.11\overline{0011}$$

$$x_3 = f^3(x_0) = 0.1\overline{0011}$$

$$x_4 = f^4(x_0) = 0.\overline{0011} \longrightarrow x_0$$

$$x_5 = f^5(x_0) = \qquad \longrightarrow x_1$$

Successive iterates repeat after x_4.

$x_0 = \frac{1}{5}$ leads to periodic orbit of period-4.

Note that, every successive iteration corresponds to previous value shifted by one symbol.

This can be generalised as

$$\boxed{S(0.b_1 b_2 b_3 b_4 ---) = 0.b_2 b_3 b_4 ----}$$

This defines a Bernoulli shift map or just shift map.

$S \rightarrow$ shift operator or map.

$b_i \rightarrow$ symbols.

Shift map is defined on a space of symbols.

- Doubling map is an example of shift map in the space of binary numbers.

Doubling map is chaotic. In this case,

chaos $\rightarrow$ loss of information, at the rate of 1 bit / iteration.

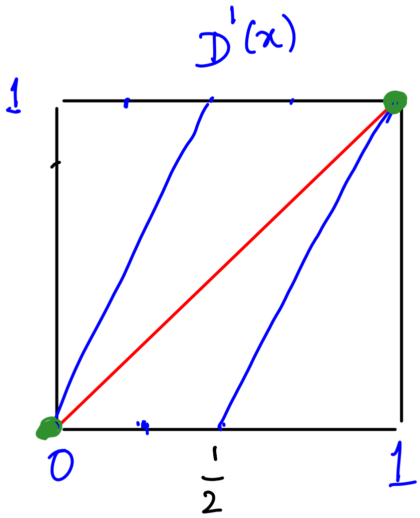
Based on the example given above, it is possible to decide which initial conditions will lead to periodic orbit.

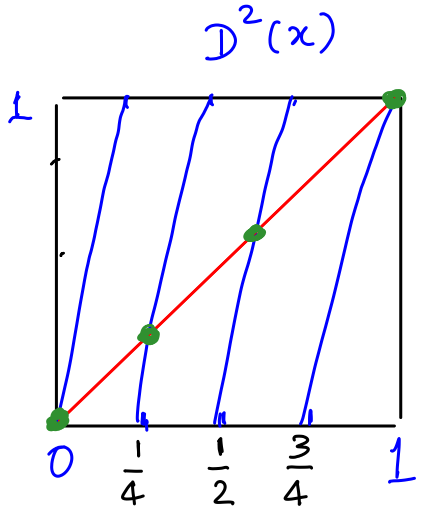
If $x_0 \rightarrow$ rational number $\rightarrow$ periodic orbit

$x_0 \rightarrow$ irrational number $\rightarrow$ not a periodic orbit (aperiodic)

If D represents the doubling map, the periodic orbits will occur if $x = D^p(x)$.

Example: $p=2$ (period-2 orbits)





—— $y=x$ line.

Fixed points $\{0\}$ $\{0, \frac{1}{3}, \frac{2}{3}\}$

What can we infer?

$p=1$: One fixed point. One period-1 orbit.

$p=2$: 3 fixed points of period-2.

Only 2 distinct fixed points.

No. of periodic orbits $= \frac{2}{2} = 1$.

$p=4$: 2^4-1 fixed points of period-4.

$x=0$ is f.p of $D^1(x)$.

$x = \frac{1}{3}, \frac{2}{3}$ are fixed points of $D^2(x)$.

$\therefore$ $15-3 = 12$ distinct fixed points.

No. of periodic orbits $= \frac{12}{4} = 3$.

period p:

In general, there are 2^p intersections of $y=x$ line with $D(x)$.

So, $2^p - 1$ distinct initial conditions which will repeat after p iterations.

As $p \to \infty$, there are infinite periodic orbits.

In general, periodic orbits form countably infinite set

But, the set of all points in $[0,1]$ forms an uncountably infinite set.

Thus, periodic points are dense in $[0,1]$, but still smaller than the number of points in $[0,1]$.

Hence, randomly chosen initial condition x_0 will not lead to a periodic orbit.

Periodic orbits are not typical, but aperiodic orbits are typical.

Logistic map and shift map are conjugate

$L: \quad x_{n+1} = 4x_n(1-x_n)$ → This is chaotic.

Let $x = \frac{1-\cos(\pi\theta)}{2} \quad (0 \leq \theta \leq 1)$

Then, we have (after substitution),

$$\cos(\pi\theta_{n+1}) = \cos(2\pi\theta_n)$$

This will be satisfied if $\theta_{n+1} = 2\theta_n \pmod 1$

This is just the Bernoulli Shift map.