Feigenbaum scaling

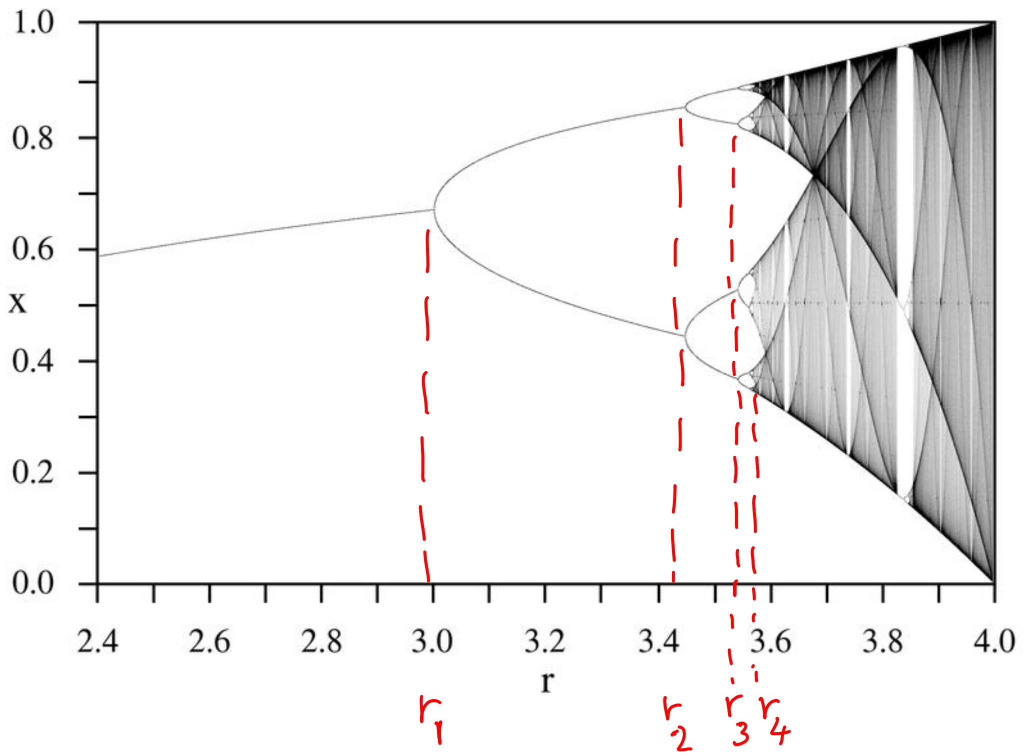
Consider 1D maps of the form

$$x_{n+1} = f(x_n, r) \qquad (r \to \text{parameter})$$

where $f(x_n)$ is a single-humped function.

Example: logistic map.

Let the map f exhibit period doubling sequence (see bifurcation diagram below).



i	r_i	Period
0	$r_0 = 1$	1
1	$r_1 = 3$	2
2	$r_2 = 3.449$	4
3	$r_3 = 3.544\ldots$	8
4	$r_4 = 3.5644$	16
⋮		
∞	$r_\infty = 3.5699$	∞

For period-2 cycle, convergence to r_∞ is geometric in nature. Convergence rate is δ.

$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

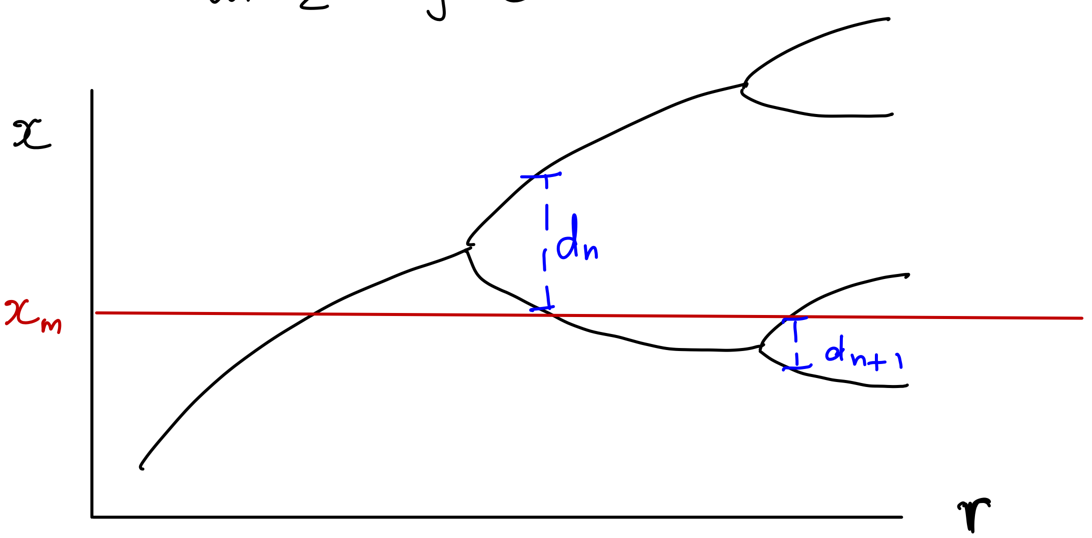
Feigenbaum number $\qquad \delta = 4.669\ldots$

Valid for all unimodal maps. It is universal.

There is one more Feigenbaum number.

$x_m \to$ value of x at which f is maximum.

$d_n \to$ distance from x_m to the nearest point in 2^n-cycle.



One more Feigenbaum number $\quad \alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = -2.5029\ldots$

Both these Feigenbaum numbers are universal. Valid for all unimodal maps.

Explaining the Feigenbaum number α

Superstable periodic cycle

Recall Lyapunov exponent

$$\lambda = \underset{n\to\infty}{Lt} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

For period-p cycle

$$\lambda = \frac{1}{p} \sum_{i=0}^{n-1} \ln |f'(x_i)| = \frac{1}{p} \ln |f^{p'}(x_0)|$$

Last equality obtained by using chain-rule in reverse.

Since it is a periodic cycle $\lambda < 0$.
Recall that $\lambda > 0$ corresponds to chaotic dynamics.

A periodic cycle is superstable if

$$|f^{p'}(x_0)| = 0 .$$

Then, $\lambda = \frac{1}{p} \ln 0 = -\infty .$

For superstable cycle, Lyapunov exponent is

$$\lambda = -\infty .$$

Perturbations decay fastest for superstable fixed points.

Perturbations are amplified for $\lambda > 0$.

Example:

$$x_{n+1} = f(x_n, r) = r - x_n^2$$

At $r = R_0$, map has superstable fixed point.

Condition for superstability $\Big\}$ $\lambda = \left.\dfrac{\partial f}{\partial x}\right|_{x=x^*} = 0$

$$\Rightarrow \left.\frac{df}{dx}\right|_{x=x^*} = -2x\Big|_{x^*} = -2x^* = 0$$

$\lambda = 0$ implies f.p is maximum of f.

At $r = R_1$, map has superstable period-2 cycle.

Let p and q denote the points of the cycle.

Condition for superstability $\Big\}$ $\lambda = (-2p)(-2q) = 0$

i.e, either $p=0$ or $q=0$.

$x=0$ is one of the points in the period-2 cycle.

period-2 condition $\Big\}$ $f^2(x) = x \Rightarrow f^2(0, R_1) = 0$

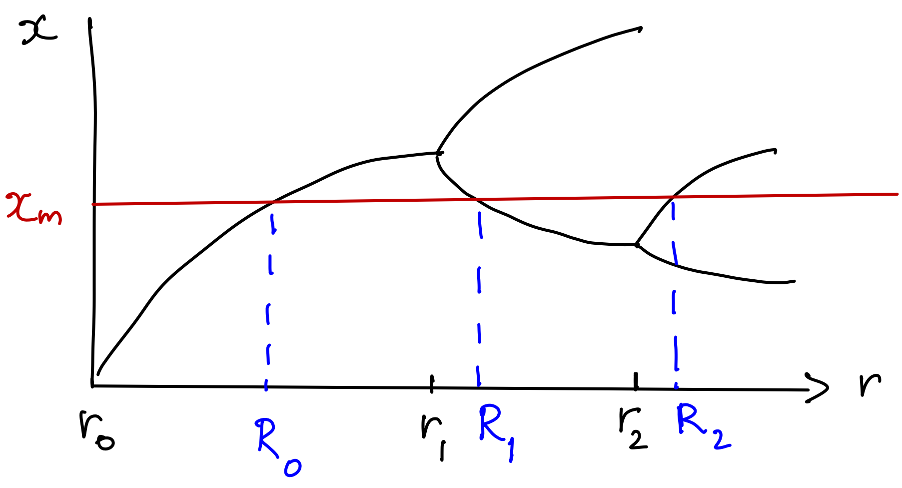
$$\Rightarrow x^*_{n+2} = r - r^2 - x_n^{*4} - 2r x_n^{*2}$$

$x^* = 0 \Rightarrow r - r^2 = 0 \Rightarrow R_1 - R_1^2 = 0$

$\Rightarrow R_1 = 0$ or $R_1 = 1$.

General rule: Supercycle of unimodal maps always contain x_m as one of its points.

How to find R_n. A graphical method:



Spacing between successive R_n shrinks by δ.

Bifurcation at r_1 $\rightarrow$ superstability at R_1 $\rightarrow$ loss of stability at r_2

This cycle repeats again and again.

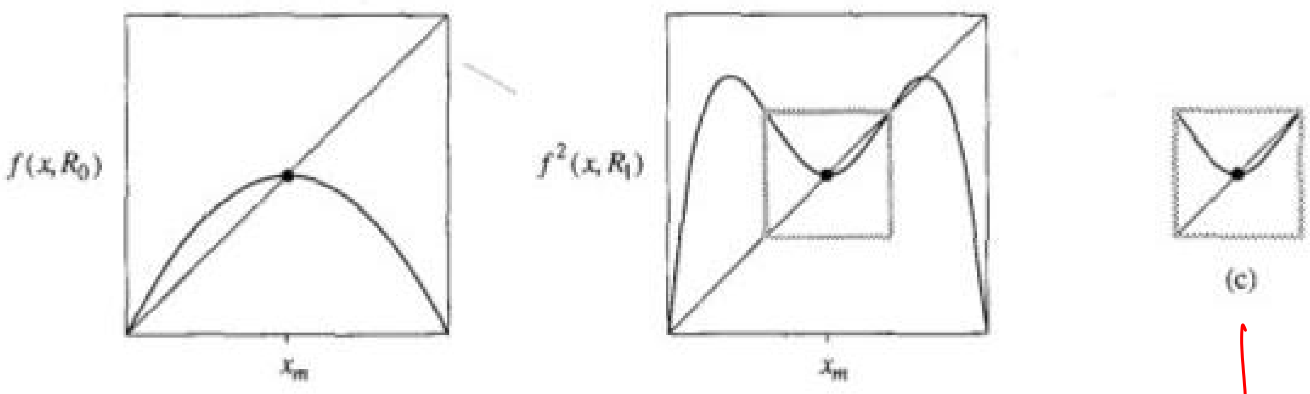
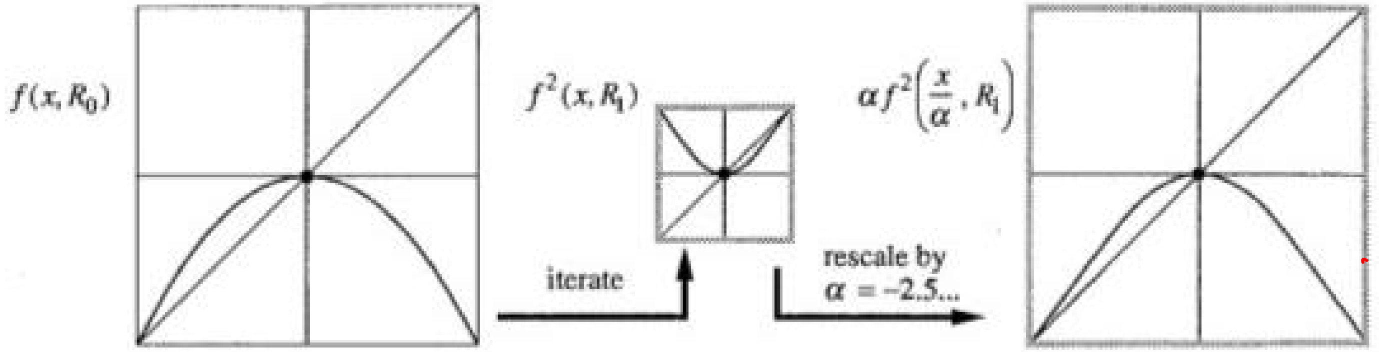


Figure 10.7.2 (from Strogatz)

Looks similar to left-most figure except for some transformations.

For convenience, redefine

$$x - x_m \rightarrow x$$



Take the middle figure. Blow it up by a factor $|\alpha| > 1$ in both the directions.
Also let $(x, y) \rightarrow (-x, -y)$.

$$f^2(x, R_1) \text{ transformed to } \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$$

Then, $$f(x, R_0) = \alpha f^2\left(\frac{x}{\alpha}, R_1\right) \qquad (\alpha < 0)$$

f has been renormalised!

This process can continue, and gives

$$f(x, R_0) = \alpha^2 f^4\left(\frac{x}{\alpha^2}, R_2\right)$$
$$= \alpha^3 f^8\left(\frac{x}{\alpha^3}, R_3\right)$$
$$\vdots$$

After renormalising n times $\Big\}$ $$f(x, R_0) = \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right)$$

As $n \to \infty$ $$\lim_{n\to\infty} \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right) = g_0(x)$$

$g_0(x) \rightarrow$ Universal limiting function

Physically, we are focussing on f^{2^n} map but only in the vicinity of $y=0$. In this region, it is a quadratic function. The quadratic region shrinks as $n \to \infty$.

$g_0(x) \to$ implies that it is independent of f.

If f has a quadratic maximum, then $g_0(x)$ is universal for all such maps.

If f has a quartic maximum, even then, a universal function exists, but it is different from $g_0(x)$.

In general,

$$g_i(x) = \lim_{n\to\infty} \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_{n+i}\right)$$

$g_i(x)$ is the universal function with superstable 2^i cycle.

Example: $i=2$, 4-cycle $\Rightarrow$ 4, 16, 64, 256 ----

$$\left.\begin{array}{l}\text{univ.}\\ \text{function}\end{array}\right\} g_4(x) = \lim_{n\to\infty} \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_{n+4}\right)$$

If we consider $i=\infty$, we have

$$f(x, R_\infty) = \alpha f^2\left(\frac{x}{\alpha}, R_\infty\right)$$

At $i=\infty$, shifting R is unnecessary.

$$g_\infty(x) = \boxed{g(x) = \alpha\, g^2\left(\frac{x}{\alpha}\right)}$$

$\downarrow$

Functional equation for $g(x)$.

We must specify boundary conditions.

Maxima of g occurs at $x=0$

$\therefore \quad g'(0) = 0 \quad$ and $\quad g(0) = 1$.

To obtain α:

Behaviour of f near $x=0$ is quadratic.

$$g(x) = a - b\,x^2$$

$a, b \rightarrow$ parameters to be determined.

Substituting this in functional equation

$$a - b\,x^2 = \alpha\left[a - b\left\{a - b\,\frac{x^2}{\alpha^2}\right\}^2\right]$$

$$= \alpha\left[a - b\,a^2\right] + \frac{2ab^2}{\alpha}\,x^2 + \underbrace{O(x^4)}_{\text{ignored}}$$

Equating coefficients,

$$a = \alpha\,(a - b\,a^2)$$

$$-a = \frac{\alpha}{2b}$$

These two equations give a quadratic equation for α :

$$\alpha^2 + 2\alpha - 2 = 0$$

$$\Rightarrow \quad \alpha = -1 \pm \frac{\sqrt{12}}{2} = -1 \pm \sqrt{3}$$

$$\alpha \simeq -2.73\ldots$$

Correct value of α : $2.502\ldots$

Less than 10% error.

Finding δ is a more elaborate exercise. We will skip this part.

$$\delta \simeq -\alpha(1-\alpha)$$