## Defining chaos

Often, physicists define chaos in terms of sensitivity to initial conditions. This property alone could be taken as the indicator of chaos. See, for example, text books by
1) Robert Hilborn, 2) Lakshmanan & Sahadevan,
3) Edward Ott.

In mathematical literature, definition of chaos is more involved. In any case, several different definitions exist.

We will adopt the following definition due to Hirsch-Smale-Devaney (often called the Devaney's definition of chaos).

## Hirsch-Smale-Devaney definition of chaos

Let $I \in [\alpha, \beta]$ be a closed interval, and $f : I \to I$ denote a mapping. Then, $f$ is said to be chaotic if

1) Periodic points of $f$ are dense in $I$,
2) $f$ is transitive in $I$,
3) $f$ has sensitive dependence on $I$.

Some explanations:

**DENSE:** Let $A$ be a subset of $B$. The set $A$ is dense in $B$ if arbitrarily close to each point in $B$, there is a point of $A$. In other words, for each point $x$ in $B$ and each $\varepsilon > 0$, the neighbourhood $N_\varepsilon(x)$ contains a point in $A$.

**TRANSITIVE:** Consider any two sub-intervals $U_1$ and $U_2$ in $I$. The map $f$ is said to be transitive in $I$, if there is a point $x_0 \in U_1$ and $n > 0$ such that $f^n(x_0) \in U_2$.

**SENSITIVE DEPENDENCE:** Consider the map $f : I \to I$. Let $x_0 \in I$ and $y_0 \in U$, where $U$ is an open interval about $x_0$. Then, for $n > 0$, if

$$|f^n(x_0) - f^n(y_0)| > c,$$

where $c$ is arbitrary real number, then $f$ is said to be sensitively dependent on $I$.

NOTE: It is shown that conditions (1) and (2) imply the condition (3).
If you are interested in this proof (though not necessary for this course), you should see:
J. Banks, J. Brooks, G. Cairns, P. Davies, P. Stacey, American Mathematical Monthly, <u>99</u> 332 (1992).

In simple terms, a system is said to be chaotic if it has bounded state space, an infinite periodic orbits that are all unstable, and exhibits exponential sensitivity to initial conditions.

Additional reading:

(For those interested, but not necessary for the course)

- Defining chaos
  M. Martelli, M. Dang and T. Seph
  Mathematics Magazine, <u>71</u> 112 (1998)