Symbolic dynamics, counting perodic orbits and chaos

Recall Bernoulli shift map (or Doubling map)

$$D: \quad \theta_{n+1} = 2\theta_n \pmod 1$$

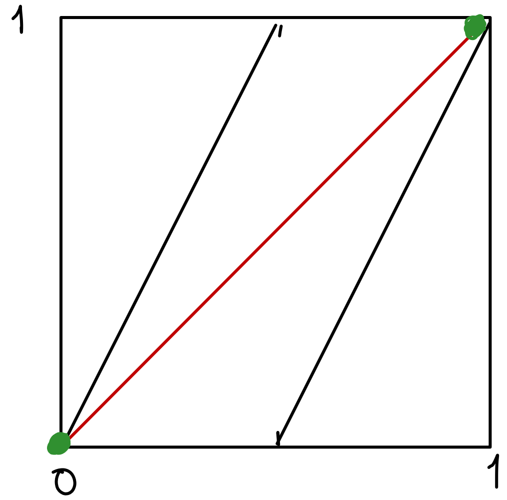
Periodic orbits were enumerated in the last class. This provides explicit calculations.

Recall that if θ is written in binary, every iteration leads to shift of one binary digit to the left. (Information loss).

Periodic orbits of period-p

Number of periodic points of period-p is equal to the number of binary sequences created from p binary digits.

$p = 1$ binary sequences of 1 bit: 0 and 1



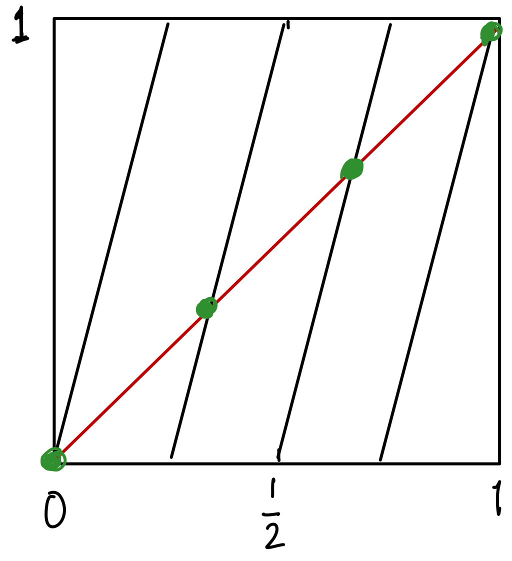
Periodic points in binary are 0.0 and 1.0

2 fixed points of period-1.

● → fixed points ——— $y = x$ line

$p=2$ binary sequence $(00, 01, 10, 11)$

Periodic points are $(0.\overline{00},\ 0.\overline{01},\ 0.\overline{10}$ and $0.\overline{11})$.



In fractional form, this is equivalent to

$$\left(0, \frac{1}{3}, \frac{2}{3}, 1\right)$$

In this, 0 and 1 are period-1 fixed points. At this level, 2 new fixed points of $p=2$.

- Verify using shift map that $1/3$ and $2/3$ are indeed period-2 orbits.

$p=3$

binary sequence : $(000, 001, 010, 100, 101, 110, 011)$

periodic points : $(0.\overline{0},\ 0.\overline{001},\ 0.\overline{010},\ 0.\overline{100},\ 0.\overline{101},\ 0.\overline{110},\ 0.\overline{011})$

In fractional form : $\left(0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\right)$

In this, 0 and 1 are same as the period-1 orbits (which can also be regarded as period-3 orbits).

6 other points are new fixed points at this level.

By using shift map, we can identify two distinct period-3 orbits. They are,

$$\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right) \quad \text{and} \quad \left(\frac{3}{7}, \frac{6}{7}, \frac{5}{7}\right).$$

Hence, 2 period-3 orbits.

$p=4$

- Identify the binary sequence with 4 digits.

Periodic points : $\Big(0.\overline{0},\ 0.\overline{0001},\ 0.\overline{0010},\ 0.\overline{0011},$
$0.\overline{0100},\ 0.\overline{0101},\ 0.\overline{0110},\ 0.\overline{0111},$
$0.\overline{1000},\ 0.\overline{1001},\ 0.\overline{1010},\ 0.\overline{1011},$
$0.\overline{1100},\ 0.\overline{1101},\ 0.\overline{1110},\ 1.\overline{0}\Big)$

In fractional form

$$\left(0, \frac{1}{15}, \frac{2}{15}, \frac{1}{5}, \frac{4}{15}, \frac{1}{3}, \frac{2}{5}, \frac{7}{15},\right.$$

$$\left.\frac{8}{15}, \frac{3}{5}, \frac{2}{3}, \frac{11}{15}, \frac{4}{5}, \frac{13}{15}, \frac{14}{15}, 1\right)$$

Using the shift map, we can identify the following orbits

$$\left.\begin{array}{l} \left(\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15}\right) \\ \left(\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}\right) \\ \left(\frac{7}{15}, \frac{14}{15}, \frac{13}{15}, \frac{11}{15}\right) \end{array}\right\} \text{3 period-4 orbits}$$

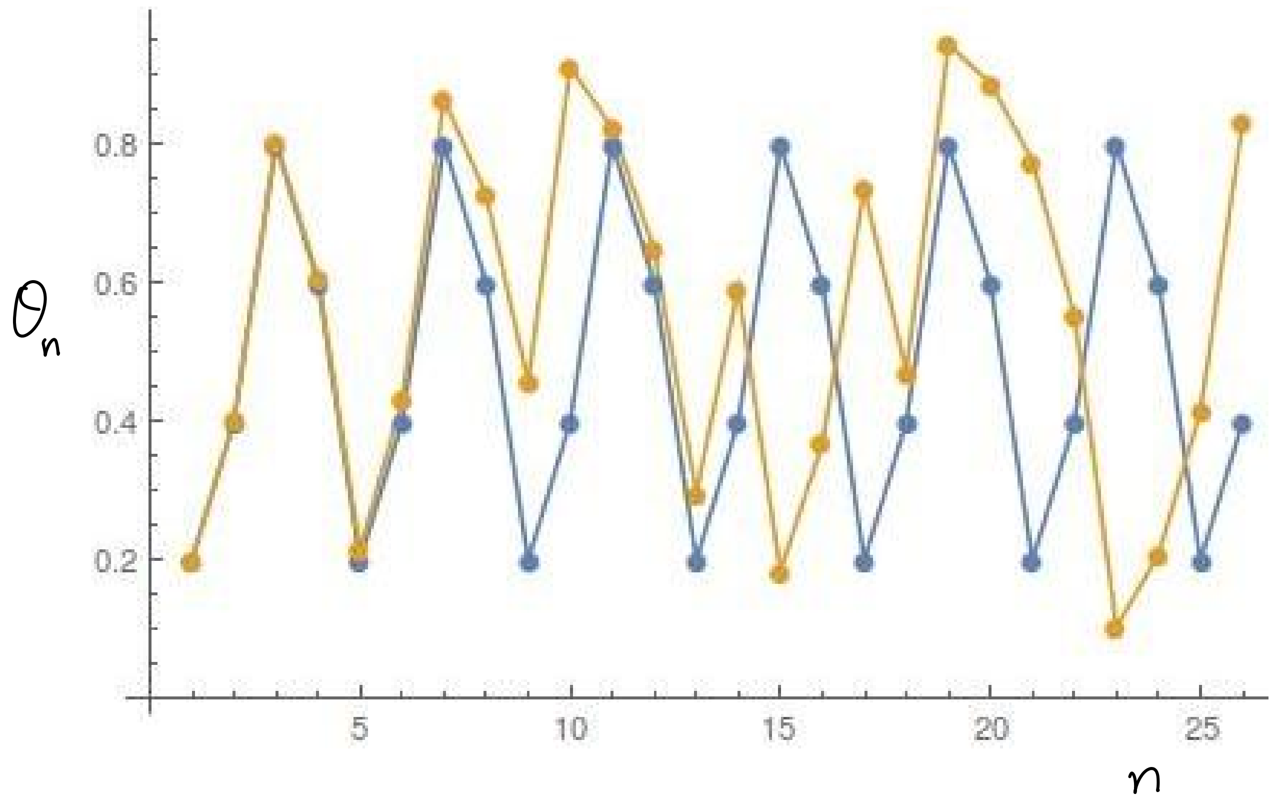
$(0, 1)$, $\left(\frac{1}{3}, \frac{2}{3}\right) \longrightarrow$ period-2 orbit identified earlier

period-1 orbit identified earlier.

12 periodic points corresponding to 3 periodic orbits.

3 periodic points inherited from $p=2$ and $p=1$.

Two orbits of shift map with close initial conditions



Two orbits started from initial conditions; $\theta_0 = \frac{1}{5}$ (blue) and

$\theta_0 = 0.201$ (orange)

Difference between the initial conditions is $\sim 10^{-3}$.

$\theta_0 = \frac{1}{5}$ corresponds to period-4 orbit (blue)

$\theta_0 = 0.201$ leads to aperiodic orbit (orange).

Shift map is chaotic. All period-p orbits are unstable.

Because slope of map is 2^p everywhere.

Shift map and logistic map

$$L: \quad x_{n+1} = 4x_n(1-x_n)$$

Shift map and logistic map are related by $x_n = \sin^2(2\pi\theta_n)$

Using this relation and the list of period-p orbits of shift map (we enumerated earlier), all the periodic orbits of logistic map can be generated and counted.

Example:

Shift map orbit with $\theta_0 = \frac{2}{15}$ is a period-4 orbit.

This transforms to $x_0 = 0.55226423$ for logistic map. This value of x_0 generates period-4 orbit for logistic map.

To exactly observe period-4 in logistic map, x_0 must be specified with infinite precision.

Note that for shift $\theta_0 = \frac{2}{15}$ is exact initial condition specified with infinite precision.

Orbits are jumbled ....

For every periodic orbit of shift map, there is one in logistic map.

However, the order in which they appear in shift map and logistic map is not the same.

Example:

In shift map $(1/3, 2/3)$ is a period-2 orbit.

If these two numbers are translated for logistic map, both give,

$$4\left(\frac{1}{3}\right)\left(1-\frac{1}{3}\right) \quad \text{and} \quad 4\left(\frac{2}{3}\right)\left(1-\frac{2}{3}\right).$$

Both are equal to $3/4$, a period-1 orbit in logistic map.

Period-2 orbit in logistic map is generated by shift map orbit $\left(\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}\right)$.

- Make a list of all orbits up to period-4 for logistic map and shift map. Compare the orbits in both the maps to realise this jumbled order.

- How many period-10 orbits are there in shift map.